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On surface waves in a finitely deformed coated half-space

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ABSTRACT

In this paper, the wave propagation in a finitely pre-deformed elastic half-space overlain by a thin coating layer (or surface film) is considered. The coated half-space is subjected to a particular uniform pre-deformation such that it is kept to be tractions-free on its surface. The first-order effective boundary conditions are introduced to approximate the effect of the overlying surface film. Then the Stroh formalism and Barnett-Lothe theory are adopted to study the surface wave characteristics. In particular, general criteria are established to identify the existence of surface waves of different modes by taking advantage of the surface impedance matrix. As an illustration, surface waves in a coated soft half-space under biasing field are investigated. Both the surface film and the half-space are modeled by the Hadamard strain energy function for soft isotropic materials. Explicit conditions for the existence of different surface wave modes (including the first-order Rayleigh waves, second-order Rayleigh waves and Love waves) are obtained, and the corresponding wavenumber ranges are also determined. Our theoretical analysis and numerical simulations show that both the surface film and the pre-deformation could remarkably affect the propagation of surface waves as well as the stability of the coated elastic half-space. Particularly, it is proved that, distinguishing from Rayleigh waves, the velocity of Love waves varies linearly with pre-stretch for a given frequency, a striking feature which is highly desirable in the sensor designs.

1. Introduction

Guided wave propagation in an elastic substrate coated with a thin surface layer has been investigated intensively by researchers with diverse applications in earth science, mechanical engineering, and solid state physics, to name a few. The substrate could be modeled as a semi-infinite medium if it is much thicker than the wavelength of interest. Even so, the wave-motion problem of an elastic half-space with a coated surface layer is still more complicated than that of a homogeneous half-space, and hence it has always been a subject of research interest (Brekhovskikh and Godin, 1990; Nayfeh, 1995; Wang et al., 2005). For example, by applying the Stroh formalism (Stroh, 1958, 1962), Darinskii (1998) systematically and theoretically investigated the leaky waves in an anisotropic layered half-space.

If the coating film is very thin as compared to the wavelength of interest, its effect on the wave motion in the coated half-space can be approximated by introducing the so-called effective boundary conditions for the half-space, which can greatly simplify the problem. The elastic layer/film could be approximately modeled by the first-order effective boundary conditions (Mindlin, 1963; Tiersten, 1969; Rokhlin and Wang, 1991; Bövik, 1994), the second-order ones (Niklasson et al., 2000a, 2000b), or even the ones of an arbitrary order (Ting, 2007). As an extension to the elastic case, Johansson and Niklasson (2003) obtained the first- and second-order effective boundary conditions for a coated piezoelectric half-space. This simplified film-substrate model can also be utilized to characterize the surface/interface effect. For example, Gurtin and Murdoch's surface elasticity theory (Gurtin and Murdoch, 1975) with zero residual surface tension (denoted as the GM theory hereafter) can be directly obtained from the effective boundary conditions (denoted as the MT conditions hereafter) (Mindlin, 1963; Tiersten, 1969) by appropriately defining the surface parameters in the GM theory. In a similar way, Chen (2011) established the theory of surface piezoelectricity by making use of the power series expansion of the transfer matrix, which correlates the state variables on the upper surface with those on the lower surface of the surface layer. Recently, Chen et al. (2014) further obtained the effective boundary conditions for a cylinder and explored the wave propagation behavior in a transversely isotropic elastic cylinder with surface effect.

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Environmental factors, such as temperature variation, mechanical constraints, and external electromagnetic field, could significantly affect the properties and functions of structures or devices. For example, temperature change or mounting force may cause deformation and stress in a piezoelectric resonator and make the resonant frequency deviate from its designed value (Tiersten and Sinha, 1979). The performance of underwater sonars or hydrophone transducers depends sensitively on the hydrostatic pressure (Wilson, 1988). On the other hand, environmental factors can be varied purposedly so that certain desired functionality or performance of a structure/device can be achieved. For example, soft acoustic devices, which have attracted considerable interest in recent years, are highly tunable by applying biasing fields since they can undergo large deformation in reversible manners under low external stimuli (Su et al., 2016). Specifically, finite deformation induced by external loading has been used to actively control the static and dynamic responses of soft metamaterials (Bertoldi et al., 2008; Galich et al., 2017). Moreover, the change in environmental factors may be detected quantitatively by means of guided waves if their dependence on environmental parameters is known. For example, the guided circumferential waves were explored to self-sense the soft electroactive cylindrical devices under biasing field (Wu et al., 2017). Thus, investigation of wave motions in elastic media under biasing fields is pivotal to many important applications. For a coated elastic half-space, under the assumption of plane-strain deformation, Steigmann and Ogden modeled a deformed thin surface film as an extensible rod and obtained the effective boundary conditions in the static (Steigmann and Ogden, 1997) and dynamic (Ogden and Steigmann, 2002) cases, respectively, for the underlying half-space. Note that, in the dynamic case, both the bending stiffness and the rotatory inertia were further considered to investigate the propagation of Rayleigh waves. While these studies are very valuable, the procedure presented by Ogden and Steigmann is slightly complicated without giving a detailed discussion on the existence and mode multiplicity of surface waves. Here, by existence, we mean that there exists a subsonic surface wave whose velocity is smaller than the so-called 'limiting speed' v_L (Chadwick and Smith, 1977), which will be defined later in this paper.

In the present work, the general theory of incremental elastic motions superimposed on a finite deformation (Dorfmann and Ogden, 2005, 2010) is employed to investigate the effect of arbitrary finite biasing fields on the propagation behavior of surface waves in a coated elastic half-space. The effective boundary conditions which approximate the effect of a thin surface film under uniform pre-deformation are obtained. Based on the Stroh formalism (Stroh, 1958, 1962) and the Barnett-Lothe theory (Barnett and Lothe, 1974, 1985; Lothe and Barnett, 1976), the equation governing surface waves in a deformed coated half-space is derived in matrix form. Then, by using the properties of the surface impedance matrix (Ingebrigtsen and Tonning, 1969; Barnett and Lothe, 1985), the criteria for the existence of various surface waves are established. The wave motion in an equi-biaxially pre-stretched isotropic elastic half-space with coating film is investigated both theoretically and numerically. The explicit conditions for the existence of various surface waves, including the first-order Rayleigh waves, second-order Rayleigh waves (also called as Sezawa waves (Sezawa and Kanai, 1935; Tiersten, 1969)), and Love waves, are established. The second-order Rayleigh waves are of significant importance since they have a larger output signal amplitude, higher propagation velocity, and larger electro-mechanical coupling parameter (for piezoelectric materials) as compared to the classical Rayleigh waves (Oliver and Ewing, 1957; Elliott et al., 1978; Emanetoglu et al., 2004; Wang et al., 2006a, 2006b; Du et al., 2008).

This paper is organized as follows. In Section 2, we introduce the general theory of an incremental elastic motion superimposed on a finite deformation. In Section 3, we obtain the characteristic dispersion equation for infinitesimal surface waves propagating in a deformed coated half-space. The criteria for the existence of surface waves are established. Then, the wave motion in a coated half-space made of isotropic Hadamard material is investigated in Section 4. The explicit conditions for the existence of surface waves of different modes and the corresponding wavenumber ranges are derived. Numerical examples are presented in Section 5, showing that both the biasing fields and the coating film could significantly affect the behavior of the surface waves. Results also indicate that the coating film could be utilized to tune the stability of the half-space. Conclusions are drawn in Section 6. Some supplementary materials are provided in Appendices A, B, and C, along with a list of symbols in Appendix D for easy reference.

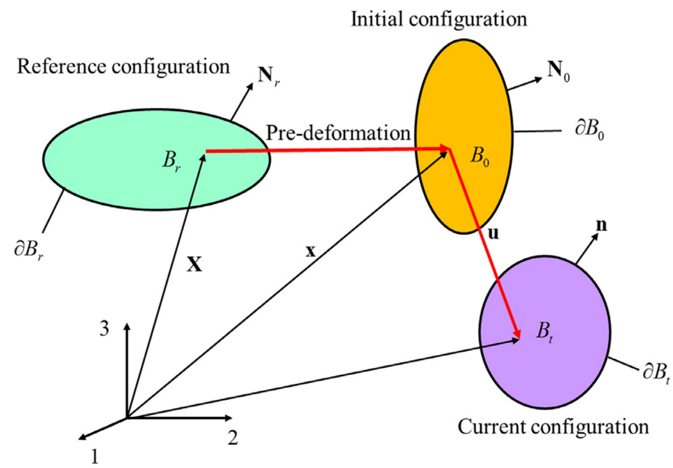

Fig. 1. Deformation of a continuous body and the three different configurations.

2. Nonlinear elasticity and theory of incremental fields – an overview

We first consider a finite static deformation of a soft elastic body, which occupies, in the three-dimensional Euclidean space, a region B_r in the undistorted reference configuration (Fig. 1). The boundary of the body in the reference configuration is denoted as ∂B_r, with its outward normal being $\mathbf{N}_r$. Let $\mathbf{X}$ be the position vector of a generic material particle in the reference configuration, which moves to a new position $\mathbf{x}$ via a smooth motion $\mathbf{x} = \mathcal{U}(\mathbf{X})$. Accordingly, the whole body deforms into the initial configuration, denoted by B_0, with boundary ∂B_0 and outward normal $\mathbf{N}_0$. The deformation gradient of the static motion is defined as $\mathbf{F} = \mathrm{Grad}\,\mathcal{U} = \partial \mathbf{x}/\partial \mathbf{X}$, and the ratio between the infinitesimal volume elements defined in the two configurations is $J = \det \mathbf{F}$. Next, we consider an infinitesimal dynamic deformation $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ superimposed on the static finite deformation $\mathbf{x} = \mathcal{U}(\mathbf{X})$. The whole body will occupy a region B_t in the current configuration, with boundary ∂B_t and the associated outward normal $\mathbf{n}$. The superimposed field is taken to be infinitesimal so that the current configuration and the initial configuration can be approximately regarded as undistinguishable.

The constitutive equations for the incremental field are (Dorfmann and Ogden, 2010)

$$\dot{K}_{0ij} = A_{0ijks} u_{s,k} \quad (i, j, k, s = 1, 2, 3), \tag{1}$$

where the subscript comma denotes differentiation with respect to the coordinate variable that follows and the superimposed dot denotes the incremental quantities; $\dot{K}_{0ij}$ is the incremental nominal stress tensor after the 'push forward' operation; A_{0ijks} is the effective elastic tensor with the major symmetry $A_{0ijks} = A_{0ksij}$, but without the minor symmetry, i.e. $A_{0ijks} \neq A_{0jiks}$. Einstein's convention for summation over repeated indices is employed throughout the paper unless stated otherwise.

The effective elastic tensor A_{0ijks} is given by (Dorfmann and Ogden, 2010)

$$A_{0ijks} = J^{-1} F_{i\alpha} F_{k\beta} \frac{\partial^2 \Omega}{\partial F_{j\alpha} \partial F_{s\beta}} \tag{2}$$

where $\Omega = \Omega(\mathbf{F})$ is the energy density function per unit volume in B_r. The effective elastic tensor is assumed to satisfy the following strong ellipticity condition (Chadwick and Smith, 1977)

$$A_{0ijks} v_i v_k w_j w_s > 0, \quad \forall \text{ nonzero real vectors } \mathbf{v}, \mathbf{w}. \tag{3}$$

In the absence of body forces, the equations of motion for the incremental field are (Dorfmann and Ogden, 2010)

$$\dot{K}_{0ij,i} = \rho u_{j,tt} \tag{4}$$

where ρ is the mass density in the current configuration. The relation between ρ and the mass density in the reference configuration, ρ_r, is $\rho = J^{-1}\rho_r$.

By defining the incremental nominal stress vectors as follows

$$\tau_1 = \begin{Bmatrix} \dot{K}_{011} \\ \dot{K}_{012} \\ \dot{K}_{013} \end{Bmatrix}, \quad \tau_2 = \begin{Bmatrix} \dot{K}_{021} \\ \dot{K}_{022} \\ \dot{K}_{023} \end{Bmatrix}, \quad \tau_3 = \begin{Bmatrix} \dot{K}_{031} \\ \dot{K}_{032} \\ \dot{K}_{033} \end{Bmatrix} \tag{5}$$

the motion Eq. (4) can be rewritten as

$$\tau_{1,1} + \tau_{2,2} + \tau_{3,3} = \rho \mathbf{u}_{,tt} \tag{6}$$

and the constitutive Eq. (1) as

$$\begin{aligned} \tau_1 &= \hat{\mathbf{C}}_1 \mathbf{u}_{,1} + \hat{\mathbf{C}}_2 \mathbf{u}_{,2} + \hat{\mathbf{C}}_3 \mathbf{u}_{,3} \\ \tau_2 &= \mathbf{C}_1 \mathbf{u}_{,1} + \mathbf{C}_2 \mathbf{u}_{,2} + \mathbf{C}_3 \mathbf{u}_{,3} \\ \tau_3 &= \breve{\mathbf{C}}_1 \mathbf{u}_{,1} + \breve{\mathbf{C}}_2 \mathbf{u}_{,2} + \breve{\mathbf{C}}_3 \mathbf{u}_{,3} \end{aligned} \tag{7}$$

where

$$\begin{aligned} &C_{1ij} = A_{02i1j}, \quad C_{2ij} = A_{02i2j}, \quad C_{3ij} = A_{02i3j} \\ &\hat{C}_{1ij} = A_{01i1j}, \quad \hat{C}_{2ij} = A_{01i2j}, \quad \hat{C}_{3ij} = A_{01i3j} \\ &\breve{C}_{1ij} = A_{03i1j}, \quad \breve{C}_{2ij} = A_{03i2j}, \quad \breve{C}_{3ij} = A_{03i3j} \end{aligned} \tag{8}$$

3. Surface waves in a coated anisotropic half-space

3.1. Stroh formalism

Let's consider the problem of wave propagation in an elastic half-space with a coating surface film, both being assumed to be arbitrarily anisotropic. As shown in Fig. A1 in Appendix A, a Cartesian coordinate system (x_1, x_2, x_3) is attached to the deformed coated half-space, with the origin located on the bottom of the coating film and the x_2-axis pointing into the half-space and perpendicular to the interface. We consider the generalized two-dimensional problems characterized by $\partial/\partial x_3 = 0$. For plane waves propagating in the x_1 direction with velocity v, the displacement field takes the following form

$$\mathbf{u} = \mathbf{a} \exp(ikz) \tag{9}$$

where $i = \sqrt{-1}$, $z = x_1 - vt + px_2$, k (≥ 0) is the wavenumber, and $\mathbf{a}$ is the unknown amplitude vector. By substituting Eq. (9) into Eq. (6) and noticing $\partial/\partial x_3 = 0$, we obtain

$$\left(\tau_1 - ik\rho v^2 \mathbf{u}\right)_{,1} + \tau_{2,2} = \mathbf{0} \tag{10}$$

Now we introduce the following stress potential vector

$$\mathbf{\Phi} = \mathbf{b} \exp(ikz) \tag{11}$$

which generates the incremental nominal stress vectors $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ through

$$\tau_2 = \mathbf{\Phi}_{,1}, \quad \tau_1 - ik\rho v^2 \mathbf{u} = -\mathbf{\Phi}_{,2} \tag{12}$$

Thus, Eq. (10) is satisfied automatically. From Eqs. (7) and (9) one obtains

$$\begin{aligned} \tau_1 - ik\rho v^2 \mathbf{u} &= ik(\mathbf{Q} + p\mathbf{R})\mathbf{a} \exp(ikz) \\ \tau_2 &= ik\left(\mathbf{R}^{\mathrm{T}} + p\mathbf{T}\right)\mathbf{a} \exp(ikz) \end{aligned} \tag{13}$$

where

$$Q_{ij} = \hat{C}_{1ij} - \rho v^2 \delta_{ij}, \quad R_{ij} = \hat{C}_{2ij}, \quad T_{ij} = C_{2ij} \tag{14}$$

and δ_{ij} is the Kronecker delta. Combination of Eqs. (11), (12) and (13) yields

$$\mathbf{b} = \left(\mathbf{R}^{\mathrm{T}} + p\mathbf{T}\right)\mathbf{a} = -p^{-1}(\mathbf{Q} + p\mathbf{R})\mathbf{a} \tag{15}$$

where the superscript "T" denotes the transpose of a matrix. Eq. (15) can be written in the following eigenrelation as

$$\mathbf{N}\xi = p\xi \tag{16}$$

where

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^{\mathrm{T}} \end{bmatrix}, \quad \xi = \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix} \tag{17}$$

are, respectively, the fundamental elastic tensor and state vector, and

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}}, \quad \mathbf{N}_2 = \mathbf{T}^{-1}, \quad \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}} - \mathbf{Q} \tag{18}$$

3.2. Barnett–Lothe theory

A new coordinate system (x'_1, x'_2, x'_3) is now introduced by rotating the old coordinate system (x_1, x_2, x_3) around the x_3-axis by an angle φ. In (x'_1, x'_2, x'_3), the eigenrelation similar to Eq. (16) can be obtained as

$$\mathbf{N}(\varphi)\xi(\varphi) = p(\varphi)\xi(\varphi) \tag{19}$$

where $\mathbf{N}(\varphi)$ is also expressed by Eqs. $(17)_1$ and (18), but $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{T}$ should be replaced by

$$\begin{aligned} \mathbf{Q}(\varphi) &= \mathbf{Q}\cos^2\varphi + \left(\mathbf{R} + \mathbf{R}^{\mathrm{T}}\right)\sin\varphi\cos\varphi + \mathbf{T}\sin^2\varphi \\ \mathbf{R}(\varphi) &= \mathbf{R}\cos^2\varphi + (\mathbf{T} - \mathbf{Q})\sin\varphi\cos\varphi - \mathbf{R}^{\mathrm{T}}\sin^2\varphi \\ \mathbf{T}(\varphi) &= \mathbf{T}\cos^2\varphi - \left(\mathbf{R} + \mathbf{R}^{\mathrm{T}}\right)\sin\varphi\cos\varphi + \mathbf{Q}\sin^2\varphi \end{aligned} \tag{20}$$

It is well known that for an undistorted elastic body in the reference configuration, $\mathbf{N}(\varphi)$ has six complex eigenvalues which occur in complex conjugate pairs as long as $0 \leq v < v_L$, where v_L is the so-called 'limiting speed' (Chadwick and Smith, 1977) defined as the minimum speed at which it is possible to find a critical angle $\varphi_L \in [0, 2\pi)$ making $\mathbf{N}(\varphi_L)$ have real eigenvalues. In the same way we can prove that for a stable elastic body under biasing field, there also exists a limiting speed v_L so that $\mathbf{N}(\varphi)$ has three pairs of conjugate complex eigenvalues provided $0 \leq v < v_L$. In the present paper, only subsonic surface waves with $0 \leq v < v_L$ are investigated, and the word 'existence' to be used later means the existence of subsonic surface waves.

We may order the six eigenvalues in a way such that $p_i(\varphi)$ ($i = 1, 2, 3$) have positive imaginary parts and $p_{i+3}(\varphi) = \bar{p}_i(\varphi)$, where the overbar denotes complex conjugate. The eigenvector associated with $p_i(\varphi)$ is denoted by $\boldsymbol{\xi}_i(\varphi)$. It was demonstrated that $\boldsymbol{\xi}_i(\varphi)$ is parallel to $\boldsymbol{\xi}_i(0)$ (Chadwick and Smith, 1977), which will be abbreviated as $\boldsymbol{\xi}_i$ in the following. Thus, $\xi_i = [\mathbf{a}_i^{\mathrm{T}} \quad \mathbf{b}_i^{\mathrm{T}}]^{\mathrm{T}}$ is also an eigenvector of $\mathbf{N}(\varphi)$. The integral matrix of $\mathbf{N}(\varphi)$ is defined as (Barnett and Lothe, 1985)

$$\tilde{\mathbf{N}} = \frac{1}{\pi}\int_0^{\pi} \mathbf{N}(\varphi)\mathrm{d}\varphi = \begin{bmatrix} \mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^{\mathrm{T}} \end{bmatrix} \tag{21}$$

where

$$\mathbf{S} = \frac{1}{\pi}\int_0^{\pi} \mathbf{N}_1(\varphi)\mathrm{d}\varphi = \mathrm{i}(2\mathbf{AB} - \mathbf{I})$$
$$\mathbf{H} = \frac{1}{\pi}\int_0^{\pi} \mathbf{N}_2(\varphi)\mathrm{d}\varphi = 2\mathrm{i}\mathbf{AA}^{\mathrm{T}}$$
$$\mathbf{L} = -\frac{1}{\pi}\int_0^{\pi} \mathbf{N}_3(\varphi)\mathrm{d}\varphi = -2\mathrm{i}\mathbf{BB}^{\mathrm{T}} \tag{22}$$

In Eq. (22), $\mathbf{I}$ is the identity matrix, and

$$\mathbf{A} = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3\end{bmatrix} \tag{23}$$

The impedance matrix $\mathbf{M}(v)$ is defined as (Barnett and Lothe, 1985)

$$\mathbf{M}(v) = -\mathrm{i}\mathbf{BA}^{-1} \tag{24}$$

Thus, the displacement polarization vector $\mathbf{a}_i$ ($i = 1, 2, 3$) and the stress polarization vector $\mathbf{b}_i$ are related by $\mathbf{b}_i = \mathrm{i}\mathbf{M}(v)\mathbf{a}_i$. The impedance matrix $\mathbf{M}(v)$ can also be expressed alternatively as (Barnett and Lothe, 1985)

$$\mathbf{M}(v) = \mathbf{H}^{-1} + \mathrm{i}\mathbf{H}^{-1}\mathbf{S} \tag{25}$$

In the following, the impedance matrix $\mathbf{M}(v)$ will be used to establish the characteristic dispersion equation for surface waves in a coated half-space. By using its properties, the criteria for the existence of surface waves then can be established.

3.3. Dispersion equation

We now consider the surface wave which propagates in the x_1-direction with its amplitude attenuating along the positive x_2-direction. The general solutions of the displacement vector and incremental stress potential vector obtained from Eqs. (9) and (11) by linear superposition of the three parts associated with p_1, p_2 and p_3 can be written as (Ting, 2007)

$$\mathbf{u} = \mathbf{A}\langle\exp(\mathrm{i}kz_*)\rangle\mathbf{q}, \quad \boldsymbol{\Phi} = \mathbf{B}\langle\exp(\mathrm{i}kz_*)\rangle\mathbf{q} \tag{26}$$

where $\mathbf{q}$ is a constant vector to be determined, and

$$\langle\exp(\mathrm{i}kz_*)\rangle = \begin{bmatrix}\exp(\mathrm{i}kz_1) & 0 & 0\\ 0 & \exp(\mathrm{i}kz_2) & 0\\ 0 & 0 & \exp(\mathrm{i}kz_3)\end{bmatrix},$$
$$z_i = x_1 + p_i x_2 - vt \quad (i = 1, 2, 3) \tag{27}$$

In Appendix A, the mechanics of a pre-deformed thin elastic film is investigated. Effective boundary conditions, i.e. Eq. (A.12), which approximate the effect of the deformed film on the substrate are established. It should be emphasized here that the direct thickness effect of the coating film has been dropped in obtaining Eq. (A12). Substitution of Eq. (26) into Eq. (A.12) yields

$$\left[\mathbf{M}(v) - hk\mathbf{N}_3^F(v)\right](\mathbf{Aq}) = \mathbf{0} \tag{28}$$

where the superscript "F" denotes the quantities of the coating film and h is its thickness after pre-deformation. To obtain Eq. (28), we have made use of Eq. (24) and

$$\mathbf{G}_1^F = \mathbf{Q}^F + \rho^F v^2\mathbf{I} - \mathbf{R}^F(\mathbf{T}^F)^{-1}(\mathbf{R}^F)^{\mathrm{T}} = -\mathbf{N}_3^F(v) + \rho^F v^2\mathbf{I}. \tag{29}$$

Because the determinant of $\mathbf{A}$ cannot be zero, for non-trivial solutions of Eq. (28), we have

$$\left|\mathbf{M}(v) - hk\mathbf{N}_3^F(v)\right| = 0 \tag{30}$$

which is the characteristic dispersion equation of surface waves in the coated half-space.

3.4. Existence and mode multiplicity of surface waves

The existence problem of surface waves in a homogeneous elastic half-space has been solved by the Stroh formalism (Stroh, 1962) and further elaborated based on the Barnett-Lothe theory (Barnett and Lothe, 1974, 1985; Lothe and Barnett, 1976). It is well known that if a surface wave exists, its mode is unique. However, for an elastic half-space with coating film, previous studies (Tiersten, 1969; Dorfmann and Ogden, 2010) have shown that the mode uniqueness of surface waves does not hold anymore, i.e. there may be multiple surface wave modes in certain appropriate situation. By adopting the surface elasticity theory (Gurtin and Murdoch, 1975), Murdoch (1976) investigated the propagation of surface waves in an isotropic elastic body with material boundary, showing that the residual stress, surface density, and surface elastic moduli all have significant effects on the surface wave behavior. In particular, in some situations there will be only one wavenumber corresponding to a fixed frequency, whilst in other situations there may be zero, two or three wavenumbers to a frequency. However, the discussion in Murdoch (1976) is somehow cumbersome, and the explicit conditions for the existence of each surface wave mode have not been established. In the following, the problem of existence and mode multiplicity of surface waves in a deformed coated half-space, will be further considered based on the dispersion Eq. (30).

As a preparation, we introduce the matrix

$$\mathbf{Z}(v,k) = \mathbf{M}(v) - hk\mathbf{N}_3^F(v) \tag{31}$$

It is well known that $\mathbf{M}(v)$ is a Hermitian matrix (Barnett and Lothe, 1985) and $\mathbf{N}_3^F(v)$ is a real symmetric matrix. Hence, it can be concluded that $\mathbf{Z}(v, k)$ is also a Hermitian matrix, and as a result, its three eigenvalues $y_1(v,k)$, $y_2(v,k)$ and $y_3(v,k)$ are all real. On differentiating $\mathbf{Z}(v, k)$ with respect to v we obtain

$$\partial\mathbf{Z}(v,k)/\partial v = \partial\mathbf{M}(v)/\partial v - hk\partial\mathbf{N}_3^F(v)/\partial v = \partial\mathbf{M}(v)/\partial v - 2hk\rho^F v\mathbf{I} \tag{32}$$

In deriving the last expression in Eq. (32), we have made use of Eqs. (14) and $(18)_3$.

For an undistorted elastic body in the reference configuration, $\partial\mathbf{M}(v)/\partial v$ in Eq. (32) is a negative definite Hemitian matrix (Barnett and Lothe, 1985; Fu and Mielke, 2002). Following Fu and Mielke (2002) (referring to Eqs. (2.21)-(2.36) there), we obtain

$$\bar{\mathbf{U}}^{\mathrm{T}}(0)\frac{\partial\mathbf{M}(v)}{\partial v}\mathbf{U}(0) = -2\rho v\int_0^{+\infty}\bar{\mathbf{U}}^{\mathrm{T}}(x_2)\mathbf{U}(x_2)\mathrm{d}x_2 \tag{33}$$

where

$$\mathbf{U}(x_2) = \mathbf{A}\langle\exp(\mathrm{i}kp_*x_2)\rangle\mathbf{q} \tag{34}$$

From Eq. (33), it can be concluded that $\bar{\mathbf{U}}^{\mathrm{T}}(0)\frac{\partial\mathbf{M}(v)}{\partial v}\mathbf{U}(0) < 0$ is valid for arbitrary nonzero vector $\mathbf{U}(0)$. Thus we conclude that $\partial\mathbf{M}(v)/\partial v$ is negative definite even if a biasing field is applied. In addition, we know from Eq. (32) that $\partial\mathbf{M}(v)/\partial v$ is a Hermitian matrix. Consequently, $\partial\mathbf{M}(v)/\partial v$ is a negative definite Hemitian matrix and its three eigenvalues $\partial y_i(v,k)/\partial v$ ($i = 1, 2, 3$), i.e. the derivatives of the eigenvalues of $\mathbf{Z}(v, k)$ defined in Eq. (31), are negative. Then, we have the following theorem.

Theorem 1. *The eigenvalues* $y_i(v,k)$ ($i = 1, 2, 3$) *of* $\mathbf{Z}(v, k)$ *are monotonically decreasing functions of the velocity* v, i.e. $\partial y_i(v,k)/\partial v < 0$.

The dispersion Eq. (30) implies that, for a given wavenumber, surface waves exist if at least one of the eigenvalues of $\mathbf{Z}(v_S,k)$ is zero, where $v_S < v_L$. Together with Theorem 1, we conclude that there are at most three surface wave modes whose dispersion relations are determined by $y_i(v_{Si},k) = 0$ ($i = 1, 2, 3$, no summation on i), respectively. We thus have the second theorem below.

Theorem 2. *If* $y_i(0,k) > 0$ *and* $y_i(v_L,k) < 0$ ($i = 1, 2, 3$)*, a subsonic velocity of the i-th order surface wave mode corresponding to the given wavenumber k can be found,* i.e. *there exists a velocity* $v_{Si} \in (0, v_L)$ *satisfying* $y_i(v_{Si},k) = 0$ ($i = 1, 2, 3$*, no summation on i*)*.*

Based on the above two theorems, we are able to investigate the mode multiplicity of surface waves, establish the criteria for the existence of surface waves, and determine the associated wavenumber ranges. In the following section, the isotropic Hadamard material model will be chosen as an example to consider the effects of coating film and biasing field on the properties of surface wave propagation.

4. Surface waves in a coated isotropic half-space of restricted Hadamard material

4.1. Propagation of surface waves

Let's consider a compressible isotropic elastic material modeled by the following Hadamard strain-energy function (Chadwick and Jarvis, 1979)

$$\Omega = \frac{1}{2}\mu\left[\mathrm{tr}\left(\mathbf{F}\mathbf{F}^{\mathrm{T}}\right) - 3\right] + \kappa f(J) \tag{35}$$

By attributing to the dimensionless response function $f(J)$ the following properties

$$f(1) = 0, \quad f'(1) = -\eta, \quad f''(1) = 1 + \frac{1}{3}\eta, \quad (\eta = \mu/\kappa) \tag{36}$$

we can interpret μ and κ respectively as the shear and bulk moduli of the material in infinitesimal deformation from reference configuration B_r. In Eq. (36), the prime denotes differentiation with respect to J. In addition to Eq. (36), $f(J)$ may be subjected to other restrictions (see Eq. (4.3) in Chadwick and Jarvis (1979)); for example, $f''(J) \geq 0$ for an arbitrary positive J.

We take the homogeneous deformation from reference configuration B_r to B_0 to be a finite equi-biaxial in-plane pre-stretch parallel to the boundary surface. We further denote λ_1 as the principal stretch along the x_1- and x_3-axes in the horizontal plane, and λ_2 as the principal stretch along the vertical x_2-axis. Substitution of Eq. (35) into Eq. (2) yields the following components of the effective elastic tensor (no summation on the repeated subscript i) (Chadwick and Jarvis, 1979)

$$A_{0ijkl} = \mu J^{-1}\lambda_i^2\delta_{ik}\delta_{jl} + \kappa\{J f'(J)\}'\delta_{ij}\delta_{kl} - \kappa f'(J)\delta_{il}\delta_{jk} \tag{37}$$

where $\lambda_3 = \lambda_1$ and $J = \lambda_1^2\lambda_2$. We assume that the coating film can be also described by the Hadamard strain-energy function (35), with μ, κ, etc. in Eq. (37) being replaced by μ^F, κ^F, etc., and $\mu^F/\kappa^F = \mu/\kappa$ (i.e. η is the same for both the film and the substrate). The primary pre-stretches of the film and substrate in the (x_1,x_3)-plane are assumed to be identical. Similar to the substrate, the effective elastic coefficients in the coating film are (again, no summation on the repeated subscript i)

$$A^F_{0ijkl} = \mu^F (J^F)^{-1}\lambda_i^2\delta_{ik}\delta_{jl} + \kappa^F[J^F f'(J^F)]'\delta_{ij}\delta_{kl} - \kappa^F f'(J^F)\delta_{il}\delta_{jk} \tag{38}$$

where $J^F = \det[\mathbf{F}^F] = \lambda_1^2\lambda_2^F$. In this paper, we further assume that the pre-deformation makes the plane of $X_2 = \text{const.}$ tractions-free in both the coating film and the substrate. Thus, we have

$$\begin{aligned} &\eta\lambda_1^{-4} + J^{-1}f'(J) = 0 \\ &\eta\lambda_1^{-4} + (J^F)^{-1}f'(J^F) = 0 \end{aligned} \tag{39}$$

From Chadwick and Jarvis (1979) we know that for an arbitrary positive value λ_1 there will be precisely one positive value of J and J^F satisfying Eq. (39), which implies $J^F = J$. This further leads to $\lambda_2^F = \lambda_2$. Thus, we are faced with the situation that the pre-stretches in the film and substrate are identical. For simplicity, we will refer to this case as the uniform pre-deformation of the coated half-space.

In the subsonic region, Chadwick and Jarvis (1979) have obtained the explicit expressions for the matrices in Eq. (22) as

$$\begin{aligned} \mathbf{S} &= [(\alpha^2 - 2\alpha\beta + 1)/(\alpha^2 - 1)](-\beta^{-1}\mathbf{e}_1\otimes\mathbf{e}_2 + \alpha^{-1}\mathbf{e}_2\otimes\mathbf{e}_1) \\ \mathbf{H} &= \mu^{-1}J\lambda_2^{-2}\Big\{[(\alpha\beta - 1)/(\alpha^2 - 1)](\beta^{-1}\mathbf{e}_1\otimes\mathbf{e}_1 + \alpha^{-1}\mathbf{e}_2\otimes\mathbf{e}_2) \\ &\quad + \alpha^{-1}\mathbf{e}_3\otimes\mathbf{e}_3\Big\} \end{aligned} \tag{40}$$

where $\mathbf{e}_i\otimes\mathbf{e}_j$ is the dyadic with $\mathbf{e}_i$ $(i = 1, 2, 3)$ being the orthogonal basis vectors, and

$$\begin{aligned} \alpha &= \lambda_2^{-1}\sqrt{\lambda_1^2 - J\mu^{-1}\rho v^2} \\ \beta &= \theta_2^{-1}\sqrt{\theta_1^2 - J\mu^{-1}\rho v^2} \\ \theta_i &= \sqrt{\lambda_i^2 + \eta^{-1}J^2 f''(J)} \quad (i = 1, 2) \end{aligned} \tag{41}$$

We then find

$$\theta_i \geq \lambda_i \quad (i = 1, 2) \tag{42}$$

due to the fact that $f''(J) \geq 0$. The limiting speed can be obtained by setting $\alpha = 0$ (Chadwick and Jarvis, 1979)

$$v_L = \sqrt{\mu/(\rho\lambda_2)} \tag{43}$$

In the subsonic region ($0 \leq v < v_L$), both α and β are real. By substituting Eq. (40) into Eq. (25) we obtain

$$\begin{aligned} \mathbf{M}(v) = \mu J^{-1}\lambda_2^2\{&[(\alpha^2 - 1)/(\alpha\beta - 1)](\beta\mathbf{e}_1\otimes\mathbf{e}_1 + \alpha\mathbf{e}_2\otimes\mathbf{e}_2) \\ &+ \alpha\mathbf{e}_3\otimes\mathbf{e}_3 - \mathrm{i}[(\alpha^2 - 2\alpha\beta + 1)/(\alpha\beta - 1)] \\ &(\mathbf{e}_1\otimes\mathbf{e}_2 - \mathbf{e}_2\otimes\mathbf{e}_1)\} \end{aligned} \tag{44}$$

It is apparent that in the subsonic region, $M_{11}(v)$, $M_{22}(v)$ and $M_{33}(v)$ are all real whilst $M_{12}(v)$ is zero or pure imaginary. By substituting the effective elastic tensors Eqs. (37) and (38) into Eq. $(18)_3$ and using Eqs. (39) and $(41)_3$ we obtain the following expression for $\mathbf{N}_3^F(v)$

$$\mathbf{N}_3^F(v) = -\mu^F J^{-1}[\lambda_1^2\mathbf{I} - \vartheta_2\lambda_2^2\mathbf{e}_1\otimes\mathbf{e}_1 - \lambda_2^2\mathbf{e}_2\otimes\mathbf{e}_2] + \rho^F v^2\mathbf{I} \tag{45}$$

where

$$\vartheta_2 = (4\lambda_2^2 - 3\theta_2^2)/\theta_2^2 \tag{46}$$

For simplicity we define $\mathbf{N}_3^0(v) = h\mathbf{N}_3^F(v)$, where h, the thickness of the deformed film, is related to the initial thickness H by $H = \lambda_2^{-1}h$. Due to the inequality (42) we obtain

$$N_{311}^0(v) - N_{322}^0(v) = -4h\mu^F J^{-1}(\theta_2^2 - \lambda_2^2)\lambda_2^2\theta_2^{-2} \leq 0 \tag{47}$$

Thus, $N_{311}^0(v) \leq N_{322}^0(v)$, and the equality holds if and only if $f''(J) = 0$.

Substitution of Eq. (44) into Eq. (28), and noticing $\mathbf{N}_3^0(v) = h\mathbf{N}_3^F(v)$, we obtain

$$\begin{bmatrix} M_{11}(v) - kN_{311}^0(v) & M_{12}(v) & 0 \\ -M_{12}(v) & M_{22}(v) - kN_{322}^0(v) & 0 \\ 0 & 0 & M_{33}(v) - kN_{333}^0(v) \end{bmatrix} (\mathbf{Aq}) = \mathbf{0} \tag{48}$$

Obviously, the surface waves polarized in the x_3-direction (the anti-plane waves) are decoupled from the surface waves polarized in the (x_1,x_2)-plane (the in-plane waves). Thus, we could rewrite the above equation as:

$$\begin{aligned} &\begin{bmatrix} M_{11}(v) - kN_{311}^0(v) & M_{12}(v) \\ -M_{12}(v) & M_{22} - kN_{322}^0(v) \end{bmatrix} \begin{Bmatrix} (\mathbf{Aq})_1 \\ (\mathbf{Aq})_2 \end{Bmatrix} = \mathbf{0} \\ &\left[M_{33}(v) - kN_{333}^0(v)\right](\mathbf{Aq})_3 = 0 \end{aligned} \tag{49}$$

The first equation is for Rayleigh waves whilst the second is for Love waves.

The dispersion equation of Rayleigh waves is thus obtained as

$$\begin{vmatrix} M_{11}(v) - kN_{311}^0(v) & M_{12}(v) \\ -M_{12}(v) & M_{22}(v) - kN_{322}^0(v) \end{vmatrix} = 0 \tag{50}$$

or

$$H_1(v)k^2 + H_2(v)k + H_3(v) = 0 \tag{51}$$

where

$$\begin{aligned} H_1(v) &= N^0_{311}(v)N^0_{322}(v) \\ H_2(v) &= -N^0_{311}(v)M_{22}(v) - N^0_{322}(v)M_{11}(v) \\ H_3(v) &= M_{11}(v)M_{22}(v) + M^2_{12}(v) \end{aligned} \tag{52}$$

From Eqs. (51) and (52), we conclude that: **1)** in the subsonic region, $H_1(v)$, $H_2(v)$ and $H_3(v)$ are all real functions; **2)** by dropping the biasing fields and appropriately defining the film parameters, the dispersion equation of Rayleigh waves in an elastic half-space with material boundary in Murdoch (1976), but in the absence of residual stress, can be obtained from Eq. (51) (see the details in Appendix B); and **3)** when the coating film is absent, we obtain $H_1(v) = H_2(v) = 0$ and Eq. (51) degenerates to

$$(\alpha^2 - 1)^{-1}[(\alpha^2 + 1)^2 - 4\alpha\beta] = 0 \tag{53}$$

which is identical to the dispersion equation of surface waves in a homogeneous elastic half-space (Chadwick and Jarvis, 1979).

The dispersion equation of Love waves derived from Eq. $(49)_2$ is

$$M_{33}(v) - kN^0_{333}(v) = 0 \tag{54}$$

It is shown in Appendix B that this equation is identical to that obtained by Murdoch (1976) in the same situation.

4.2. Existence and mode multiplicity of surface waves

The dispersion Eq. (54) of Love waves is simple, which enables us to analyze the conditions for their existence directly. On the other hand, the dispersion Eq. (51) of Rayleigh waves is much complicated, and a direct analysis may be very tedious. Thus, we will apply Theorems 1 and 2 stated in Section 3.4 to treat both Rayleigh and Love waves in a uniform way so that the conditions for their existence and the associated mode multiplicity can be obtained explicitly. This is presented below.

By substituting Eqs. (44) and (45) into Eq. (31) and solving the eigenvalue problem, we obtain the explicit expressions of $y_i(v,k)$ $(i = 1, 2, 3)$

$$\begin{aligned} y_1(v,k) &= \frac{X(v) - \sqrt{Y^2(v) - 4M^2_{12}(v)}}{2} \\ y_2(v,k) &= \frac{X(v) + \sqrt{Y^2(v) - 4M^2_{12}(v)}}{2} \\ y_3(v,k) &= M_{33}(v) - kN^0_{333}(v) \end{aligned} \tag{55}$$

where

$$\begin{aligned} X(v) &= M_{11}(v) + M_{22}(v) - k[N^0_{311}(v) + N^0_{322}(v)] \\ Y(v) &= M_{11}(v) - M_{22}(v) - k[N^0_{311}(v) - N^0_{322}(v)] \end{aligned} \tag{56}$$

We assume first that there are two subsonic Rayleigh wave velocities v_{S1} and v_{S2} satisfying $y_1(v_{S1},k) = 0$ and $y_2(v_{S2},k) = 0$. We can prove that $v_{S1} < v_{S2}$: If $y_2(v_{S2},k) = 0$, then $X(v_{S2}) < 0$. Thus, according to Eq. $(55)_1$ we conclude that $y_1(v_{S2},k) < 0$. Combining conditions $y_1(v_{S2},k) < 0$, $y_1(v_{S1},k) = 0$, and Theorem 1 in Section 3.4, we have $v_{S1} < v_{S2}$. Therefore, we have the following two corollaries by further making use of Theorem 2 in Section 3.4.

Corollary 1. *The conditions for the existence of the low-order (first-order) Rayleigh waves (or FRWs) are $y_1(0,k) > 0$ and $y_1(v_L,k) < 0$.*

Corollary 2. *The conditions for the existence of the high-order (second-order) Rayleigh waves (or SRWs) are $y_2(0,k) > 0$ and $y_2(v_L,k) < 0$.*

If there is a subsonic velocity v_{S3} satisfying $y_3(v_{S3},k) = 0$, v_{S3} is then the velocity of Love waves. Again, together with Theorem 2 in Section 3.4, we have the following corollary.

Corollary 3. *The conditions for the existence of Love waves are $y_3(0,k) > 0$ and $y_3(v_L,k) < 0$.*

Table 1
Wavenumber ranges of FRWs.

Biasing field	Coating parameter	Wavenumber range
$\lambda_1 > \lambda_2$	$\gamma < \lambda_1^{-2}(\lambda_1^2 - \lambda_2^2)$	$0 \le k < k_2(v_L)$
	$\gamma \ge \lambda_1^{-2}(\lambda_1^2 - \lambda_2^2)$	$k \ge 0$
$\lambda_1 = \lambda_2$ and $H_3(0) > 0$	Arbitrary	$k \ge 0$
$\lambda_1 = \lambda_2$ and $H_3(0) \le 0$	Arbitrary	$k > \frac{H_3(0)}{N^0_{311}(0)M_{22}(0)}$
$\lambda_1 < \lambda_2$ and $H_3(0) > 0$	Arbitrary	$0 \le k < k_1(0)$
$\lambda_1 < \lambda_2$, $H_3(0) \le 0$ and $\lambda_1^2 > \Psi\lambda_2^2$	Arbitrary	$k_2(0) < k < k_1(0)$
$\lambda_1 < \lambda_2$, $H_3(0) \le 0$ and $\lambda_1^2 \le \Psi\lambda_2^2$	Arbitrary	$k \in Ø$

Table 2
Wavenumber ranges of SRWs.

Biasing field	Coating parameter	Wave number range
$\lambda_1^2 > \vartheta_2\lambda_2^2$	$\gamma > \lambda_1^{-2}(\lambda_1^2 - \vartheta_2\lambda_2^2)$	$k > k_2(v_L)$
	$\gamma \le \lambda_1^{-2}(\lambda_1^2 - \vartheta_2\lambda_2^2)$	$k \in Ø$
$\lambda_1^2 \le \vartheta_2\lambda_2^2$	Arbitrary	$0 \le k < k_2(0) \cap k > k_2(v_L)$

The solutions of $y_i(0,k) > 0$ and $y_i(v_L,k) < 0$ $(i = 1, 2)$ in various situations are discussed in detail in Appendix C. From the derived solutions, we can identify the wavenumber ranges of FRWs and SRWs corresponding to various biasing fields and coating parameters, which are summarized in Tables 1 and 2, which also clearly indicate the conditions for the existence of each wave mode. In the two tables, symbols Ø and ∩ denote, respectively, the empty set and the intersection of sets, and the relevant parameters are defined as

$$\begin{aligned} &\gamma = v_L^2/\left(v_L^F\right)^2, \quad \Delta(v) = H_2^2(v) - 4H_1(v)H_3(v) \\ &k_1(v) = \frac{-H_2(v) - \sqrt{\Delta(v)}}{2H_1(v)}, \quad k_2(v) = \frac{-H_2(v) + \sqrt{\Delta(v)}}{2H_1(v)} \\ &\Psi = \frac{M^2_{22}(0)\vartheta_2 + \left[\sqrt{-H_3(0)} + \sqrt{-M^2_{12}(0)}\right]^2}{\left[\sqrt{-H_3(0)} + \sqrt{-M^2_{12}(0)}\right]^2 + M^2_{22}(0)} \end{aligned} \tag{57}$$

where v_L^F is the limiting speed of the coating film expressed by

$$v_L^F = \sqrt{\mu^F/(\lambda_2\rho^F)} \tag{58}$$

We further point out that, from Chadwick and Jarvis (1979) we find that $H_3(0) > 0$ is the stability condition of a homogeneous elastic half-space.

For biasing fields satisfying $\lambda_1 \le \lambda_2$, the inequality $\gamma \ge \lambda_1^{-2}(\lambda_1^2 - \lambda_2^2)$ holds automatically, and consequently the coating parameters can be arbitrary. It is proved in Appendix C that the inequality $\lambda_1 \le \lambda_2$ holds naturally if $H_3(0) \le 0$. Table 1 can be utilized to predict the wavenumber range of FRWs for the given biasing field and coating parameter. For example, if the biasing field satisfies $\lambda_1 > \lambda_2$ and the coating parameter γ satisfies $\gamma < \lambda_1^{-2}(\lambda_1^2 - \lambda_2^2)$, then FRWs exist for $0 \le k < k_2(v_L)$. Our further study reveals that FRWs will transfer into supersonic waves if $k > k_2(v_L)$. To summarize, it is shown in Table 1 that: **1)** if the biasing field and coating parameter respectively satisfy $\lambda_1 > \lambda_2$ and $\gamma \ge \lambda_1^{-2}(\lambda_1^2 - \lambda_2^2)$, then FRWs exist for $k \ge 0$; **2)** if the biasing field satisfies $\lambda_1 < \lambda_2$ and $H_3(0) > 0$, then FRWs exist for $0 \le k < k_1(0)$; and **3)** if the biasing field satisfies $\lambda_1 < \lambda_2$, $H_3(0) \le 0$ and $\lambda_1^2 > \Psi\lambda_2^2$, then FRWs exist in the region $k_2(0) < k < k_1(0)$.

Table 2 shows the wavenumber ranges of SRWs in various situations. The coating parameter can be arbitrary when $\lambda_1^2 \le \vartheta_2\lambda_2^2$ because $\gamma > \lambda_1^{-2}(\lambda_1^2 - \vartheta_2\lambda_2^2)$ holds automatically in this situation. It is shown in Table 2 that: **1)** when the biasing field satisfies $\lambda_1^2 > \vartheta_2\lambda_2^2$, then SRWs exist for $k > k_2(v_L)$ if the coating parameter satisfies $\gamma > \lambda_1^{-2}(\lambda_1^2 - \vartheta_2\lambda_2^2)$, whilst SRWs do not exist if γ satisfies $\gamma \le \lambda_1^{-2}(\lambda_1^2 - \vartheta_2\lambda_2^2)$; and **2)** when the biasing field satisfies $\lambda_1^2 \le \vartheta_2\lambda_2^2$, then the existence of SRWs depends on the values of $k_2(0)$ and $k_2(v_L)$, i.e. if $k_2(0) \le k_2(v_L)$, SRWs do not exist, while

Table 3
Wavenumber ranges of Love waves.

Coating parameter	Wave number range
$\gamma > 1$	$k \geq 0$
$\gamma \leq 1$	$k \in \emptyset$

if $k_2(0) > k_2(v_L)$, SRWs exist for $k_2(v_L) < k < k_2(0)$. These features indicate that the necessary condition for the existence of SRWs is $\gamma > \lambda_1^{-2}(\lambda_1^2 - \vartheta_2\lambda_2^2)$. Thus, it becomes possible to tune SRWs by choosing an appropriate coating film or by changing the biasing field.

For Love waves, the necessary and sufficient condition of $y_3(0,k) > 0$ is

$$M_{33}(0) - kN^0_{333}(0) > 0 \tag{59}$$

This inequality holds automatically since $M_{33}(0) = \mu\lambda_1^{-1} > 0$ and $N^0_{333}(0) = -h\mu^F J^{-1}\lambda_1^2 < 0$. As a result, $y_3(0,k)$ must be positive.

The necessary and sufficient condition of $y_3(v_L,k) < 0$ is

$$N^0_{333}(v_L) > 0 \tag{60}$$

namely

$$\gamma > 1 \tag{61}$$

As can be observed from Table 3 that the existence of Love waves depends only on the coating parameter γ. Love waves exist in the coated half-space with a soft surface film (i.e., the limiting speed of the film is lower than that in the substrate, namely $\gamma > 1$), whilst they disappear when there is a stiff coating film over the substrate (namely $\gamma \leq 1$).

As mentioned earlier, the dispersion Eq. (54) for Love waves is simple. Therefore, we are able to obtain the existence condition for Love waves directly from Eq. (54), which can be written as

$$k = \frac{J^{-1}\lambda_2^2\mu\alpha}{h\rho^F v^2 - h\lambda_2^{-1}\mu^F} \tag{62}$$

Obviously, the wavenumber is real and positive if the following two conditions are satisfied:

(1) $v^2 > \mu^F\lambda_1^2 J^{-1}/\rho^F$;
(2) α is real, which means $v^2 \leq J^{-1}\lambda_1^2\mu/\rho$.

From these two conditions, we conclude that Love waves exist if and only if

$$\mu^F/\rho^F < \mu/\rho \tag{63}$$

The inequality (63) is equivalent to the inequality (61).

5. Application to rubber materials with numerical results

As an application, we consider the continuum nearly-incompressible rubber whose response function $f(J)$ is given by (Chadwick and Jarvis, 1979)

$$f(J) = \frac{1}{m}\left(J + \frac{1}{m-1}J^{-m+1} - \frac{m}{m-1}\right) - \frac{3}{2}\eta\left(J^{2/3} - 1\right) \tag{64}$$

where the parameters are selected for the substrate as $m = 10$, $\eta = 5\times10^{-4}$, $\mu = 10^6$Pa and $\rho_r = 10^3\,\text{kgm}^{-3}$. The parameters of the coating film are related with those of the substrate by

$$\mu^F = r\mu, \quad \rho_r^F = r\gamma\rho_r \tag{65}$$

where γ is defined in Eq. $(57)_1$ and r is the shear modulus ratio. We define the dimensionless wavenumber $\tilde{k} = Hk$ and dimensionless velocity $s = v/v_b$, where $v_b = \sqrt{\mu/\rho_r}$. The thickness of the coating film in the reference configuration is taken to be $H = 10^{-7}$m.

For numerical illustration, three different biasing fields are considered, namely, B_1) $\lambda_1 = 0.8$ and $\lambda_2 = 1.56$, i.e. the coated half-space is equi-biaxially compressed in the horizontal (X_1,X_3)-plane; B_2) $\lambda_1 = \lambda_2 = 1$, which means there is no biasing field; and B_3) $\lambda_1 = 1.5$ and $\lambda_2 = 0.445$, i.e. the structure is equi-biaxially tensed in the horizontal (X_1,X_3)-plane. As assumed earlier, the pre-deformation always makes the plane of $X_2 =$ const. tractions-free. In addition, two types of coating film are considered: C_A) $r = 0.1$ and $\gamma = 100$, which implies a soft coating film ($v_L^F < v_L$); and C_B) $r = 10$ and $\gamma = 0.01$, which corresponds to a stiff coating film ($v_L^F > v_L$).

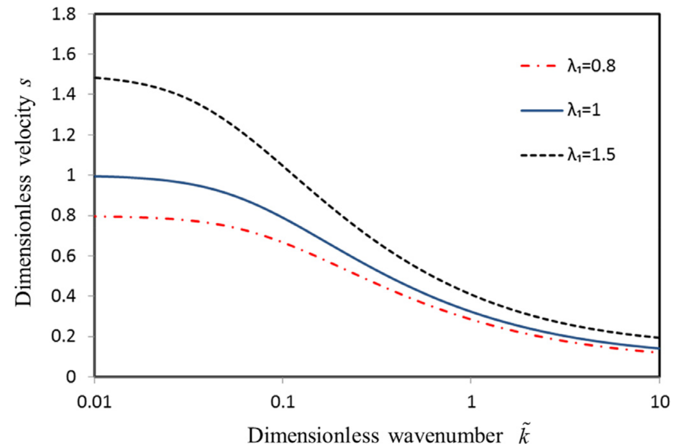

Fig. 2. Dispersion curves (dimensionless velocity vs. dimensionless wavenumber) of Love waves in an elastic half-space with a soft coating film under three kinds of biasing fields.

First, we consider Love waves. According to Eq. (61), Love waves exist if and only if $\gamma > 1$. As a result, there are no Love waves in a rubber half-space covered by a C_B–type film. On the other hand, the dispersion curves of Love waves in a rubber half-space covered by a C_A–type film under different biasing fields are plotted in Fig. 2.

It is observed from Fig. 2 that, when $\tilde{k}$ tends to zero, the dimensionless velocity s tends to v_L/v_b (i.e., the velocity v tends to the limiting speed v_L). Thus, the physical interpretation of Love waves is that a soft coating film would slow down the propagation of shear waves in the substrate and consequently concentrates the energy of shear waves near the surface. The velocity of Love waves is between the limiting speed of the coating film and that of the substrate, i.e. $v_L^F < v < v_L$, which reduces to the velocity range of Love waves obtained by Murdoch (1976) (see his Eq. (3.6)) when neglecting the biasing field. The comparison of the three curves in Fig. 2 demonstrates further that a compressive biasing field slows down the propagation of Love waves whilst a tensile biasing field accelerates their propagation. As a result, it is possible to tune Love wave devices by exerting different biasing fields.

Since the dispersion equation of Love waves is simple, they may be used to self-sense the considered structure under a biasing filed. Fig. 3 shows that, for a given frequency, the velocity of Love waves s varies linearly with pre-stretch λ_1. Such a linear function feature is particularly appealing in sensor designs. Actually from Eq. (62), the following analytical linear relation between the velocity and pre-stretch can be derived, which could be conveniently used by sensor designers.

$$s = \lambda_1\sqrt{\frac{2\gamma r^2H^2\omega^2/v_b^2 + 1 + \sqrt{4(\gamma-1)r^2H^2\omega^2/v_b^2 + 1}}{2r^2\gamma^2H^2\omega^2/v_b^2 + 2}}. \tag{66}$$

Eq. (66) demonstrates that the pre-stretch can be predicted (or monitored) accurately by measuring the velocity of Love waves. It further shows that, for a given biasing field, the thickness H of the surface thin film can be determined accurately by measuring the velocity and frequency of Love waves (also making use of Eq. (62)).

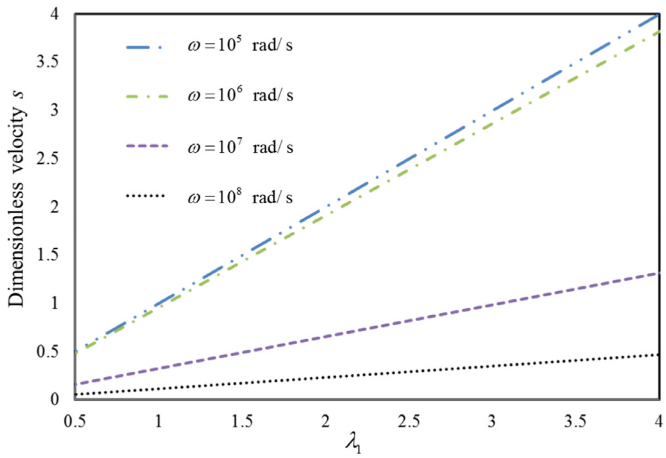

Fig. 3. Linear variation of Love wave velocity s vs. pre-stretch λ_1.

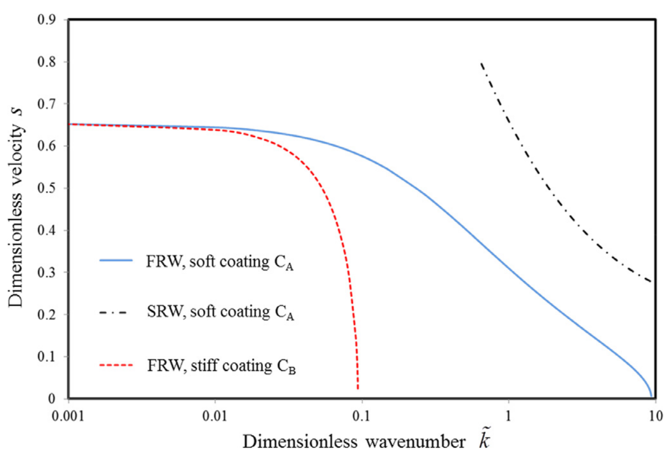

Fig. 4. Dispersion curves of Rayleigh waves in a coated rubber half-space under the biasing field in Case B_1 (equi-biaxially compressed).

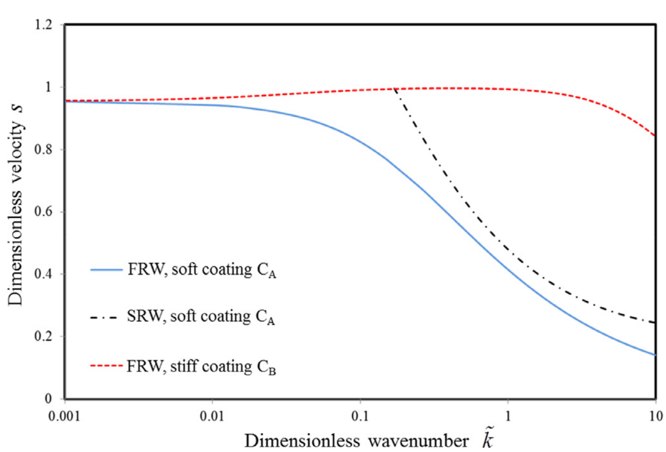

Fig. 5. Dispersion curves of Rayleigh waves in a coated rubber half-space without biasing field (i.e. Case B_2).

As for Rayleigh waves, the dispersion curves of FRWs and SRWs can be determined from Eq. (50). The results are presented in Figs. 4, 5 and 6, respectively, under the biasing field in Case B_1 (equi-biaxially compressed), Case B_2 (without biasing field), and Case B_3 (equi-biaxially tensed). In these figures, the solid line corresponds to FRWs associated with the soft film C_A, dashed line to FRWs associated with the stiff film C_B, and dot-dashed line to SRWs associated with the soft film. It is observed that the dispersion curves for Rayleigh waves under a biasing field (Figs. 4 and 6) are completely different from those without a biasing field (Fig. 5). In particular, the dispersion curves of FRWs and SRWs with a soft coating film do not intersect with each other (although the two curves in Fig. 6 are extremely close to each other).

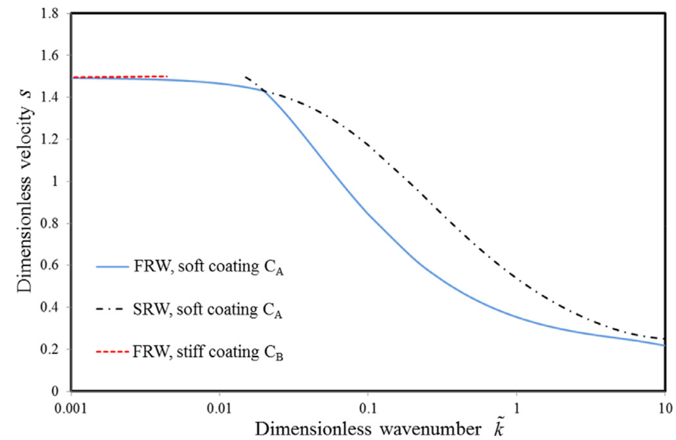

Fig. 6. Dispersion curves of Rayleigh waves in a coated rubber half-space under the biasing field in Case B_3 (equi-biaxially tensed).

To understand in more detail of the wave features, we compute the wavenumber ranges of FRWs and SRWs corresponding to various biasing fields and coating films based on Tables 1 and 2, and compare these predictions with the numerical results shown in Figs. 4-6.

First, for the coated half-space under the compressive biasing field B_1 shown in Fig. 4, we have $\lambda_1 < \lambda_2$ and $H_3(0) = 5.54\mu^2 > 0$. According to Table 1, the wavenumber range of FRWs is $0 \le \tilde{k} < \tilde{k}_1(0)$ in this situation. For the soft film C_A, we have $\tilde{k}_1(0) = 6.295$ and for the stiff film C_B we have $\tilde{k}_1(0) = 0.0629$. In addition, because $\vartheta_2\lambda_2^2 = -7.31$ for this biasing field, we obtain $\lambda_1^2 > \vartheta_2\lambda_2^2$. For the soft film C_A, we have

$$\lambda_1^{-2}(\lambda_1^2 - \vartheta_2\lambda_2^2) = 12.4 < \gamma = 100 \tag{67}$$

According to Table 2, SRWs exist in this situation with its wavenumber range $\tilde{k} > \tilde{k}_2(v_L) = 0.432$. However, for the stiff film C_B

$$\lambda_1^{-2}(\lambda_1^2 - \vartheta_2\lambda_2^2) = 12.4 > \gamma = 0.01 \tag{68}$$

Thus SRWs do not exist according to Table 2. Obviously, the predictions from Tables 1 and 2 are completely coincident with the numerical results shown in Fig. 4.

Second, for a coated half-space without biasing field (i.e. Case B_2), as shown in Fig. 5, we have

$$\lambda_1^{-2}(\lambda_1^2 - \lambda_2^2) = 0 < \gamma \tag{69}$$

According to Table 1, whether the coating film is soft (C_A) or stiff (C_B), FRWs exist in the full wavenumber region, i.e. $\tilde{k} \ge 0$. In addition, $\vartheta_2\lambda_2^2 = -3$, and therefore for the soft film C_A

$$\lambda_1^{-2}(\lambda_1^2 - \vartheta_2\lambda_2^2) = 4 < 100 \tag{70}$$

The wavenumber range of SRWs is $\tilde{k} > \tilde{k}_2(v_L) = 0.167$ according to Table 2, and for the stiff film C_B,

$$\lambda_1^{-2}(\lambda_1^2 - \vartheta_2\lambda_2^2) = 4 > 0.01 \tag{71}$$

Thus, SRWs do not exist according to Table 2. These predictions are also consistent with the numerical results presented in Fig. 5.

Based on the MT condition, Tiersten has investigated the surface waves in an undeformed layered elastic body (Tiersten, 1969). In fact, our boundary condition is identical with the MT condition when the pre-deformation is removed. He observed the existence of the first order Rayleigh waves and the Sezawa waves.

The characteristics of the dispersion curves of these two surface wave modes calculated in Tiersten (1969) are quite similar to our results in Fig. 5 for soft coating condition. He compared his approximate results calculated based on MT condition with the exact results and found that the approximated dispersion curve of the first order Rayleigh waves was accurate when $\tilde{k} < 0.2$. He also found that the dispersion curve of the Sezawa waves was correct, but quite not accurate for almost all the wavenumber range. According to the consistency between our theoretical results with those in Tiersten (1969), we can deduce the same conclusion for the accuracy of the dispersion curves calculated in this manuscript.

Finally, we look at the coated half-space under the tensile biasing field B_3 which satisfies $\lambda_1 > \lambda_2$ and $\vartheta_2\lambda_2^2 = -0.594 < \lambda_1^2$, as shown in Fig. 6. If the half-space is covered by the soft film C_A, we then have

$$\lambda_1^{-2}(\lambda_1^2 - \lambda_2^2) = 0.912 < \gamma = 100$$
$$\lambda_1^{-2}(\lambda_1^2 - \vartheta_2\lambda_2^2) = 1.264 < \gamma = 100 \quad (72)$$

Thus, according to Tables 1 and 2, the wavenumber ranges of FRWs and SRWs are respectively $\tilde{k} \geq 0$ and $\tilde{k} > \tilde{k}_2(v_L) = 0.0326$. If the half-space is covered by the stiff film C_B, we then have

$$\lambda_1^{-2}(\lambda_1^2 - \lambda_2^2) = 0.912 > \gamma = 0.01$$
$$\lambda_1^{-2}(\lambda_1^2 - \vartheta_2\lambda_2^2) = 1.264 > \gamma = 0.01 \quad (73)$$

From Table 1, the wavenumber range of FRWs is found to be $0 \leq \tilde{k} < \tilde{k}_2(v_L) = 0.01$, whilst from Table 2, SRWs do not exist. Fig. 6 shows that the velocity of FRWs corresponding to the stiff film increases monotonically with the increasing wavenumber, and reaches the limit v_L when the dimensionless wavenumber increases to $\tilde{k} = 0.01$. By further increasing the wavenumber, i.e. $\tilde{k} > 0.01$, the FRW would change to a supersonic wave. Or in other words, in terms of only the subsonic waves, FRWs do not exist when $\tilde{k} > 0.01$. Once again, the wave features shown in Fig. 6 match those predicted by the formulas in Tables 1 and 2.

The comparison among Figs. 4, 5 and 6 indicates that the compressive biasing field pushes the dispersion curves of FRWs and SRWs away from each other (Fig. 4), while the tensile biasing field pulls the two curves close to each other (Fig. 6). It is further noticed from Fig. 6 that mode switching happens at the point where the two curves become extremely close to each other. This kind of mode switching was also reported in Adler and Solie (1995) and Nakahata et al. (1995) to name a few, and can be further understood by looking at the wave polarization vector. The polarization of SRWs is $\mathbf{a} = (1 \quad -2.66i \quad 0)^T$ at $\tilde{k} = 0.043$ and $\mathbf{a} = (1 \quad 1.7i \quad 0)^T$ at $\tilde{k} = 0.045$, and the polarization of FRWs is $\mathbf{a} = (1 \quad 0.23i \quad 0)^T$ at $\tilde{k} = 0.043$ and $\mathbf{a} = (1 \quad -1.515i \quad 0)^T$ at $\tilde{k} = 0.045$. Consequently, mode switching happens at a point between $\tilde{k} = 0.043$ and $\tilde{k} = 0.045$. Therefore, by exerting an equi-biaxial tensile biasing field on a coated half-space, we may be able to induce mode switching, whilst by applying an equi-biaxial compressive biasing field the dispersion curves of the two modes could be pushed away from each other, which is particularly beneficial in the excitation of single wave mode (FRW or SRW).

The comparison between the simulated results and Tables 1–3 has illustrated the validity of the theoretical predictions in these tables. Now we use these predictions to construct a phase diagram to show the existence and mode multiplicity of surface waves in the (λ_1,γ) plane. We confine ourselves to the case of the soft coated elastic half-space subjected to an in-plane pre-stretched $\lambda_1 \geq 1$ (i.e. equi-biaxially tensed). The diagram is shown in Fig. 7, which indicates four different regions: **1)** in Region A, FRWs exist within the long wave region $0 \leq k < k_2(v_L)$, but there is no SRW or no Love wave; **2)** in Region B, FRWs exist in the full wavenumber region $k \geq 0$, but there is neither SRW nor Love wave; **3)** in

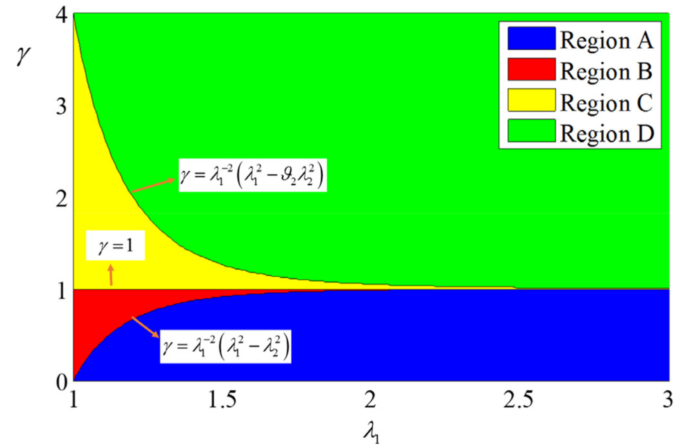

Fig. 7. Phase diagram of the existence and mode multiplicity of surface waves in the (λ_1,γ) plane.

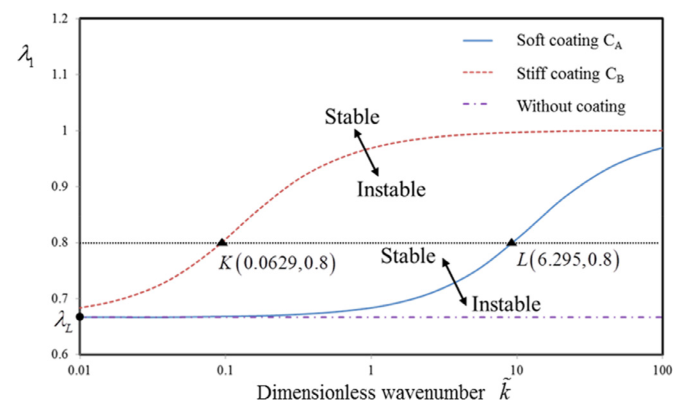

Fig. 8. Effect of the coating film on the stability of the rubber half-space.

Region C, FRWs and Love waves exist in the full wave number region $k \geq 0$, but SRWs do not exist; and **4)** in Region D, FRWs and Love waves exist in the full wave number region $k \geq 0$, while SRWs exist in the short wave region $k > k_2(v_L)$. These regions are separated by three boundary curves, i.e. $\gamma = \lambda_1^{-2}(\lambda_1^2 - \lambda_2^2)$, $\gamma = 1$, and $\gamma = \lambda_1^{-2}(\lambda_1^2 - \vartheta_2\lambda_2^2)$. As one can see from the figure, the existence of SRWs depends on the tensile biasing field. For example, for an undistorted coated half-space, SRWs exist only if the coating film is as soft as $\gamma > 4$; while for a tensile biasing field with $\lambda_2 = 1.5$, SRWs can exist even when the coating film is as stiff as $\gamma = 1.5$. Therefore, for a specified surface film, pre-deformation can be an effective means to switch on or off the propagation of SRWs. Consequently, this phase diagram could be applied directly as a guide for actively manipulating the performance of soft acoustic devices.

Region A: FRWs exist for $0 \leq k < k_2(v_L)$, SRWs or Love waves do not exist; Region B: FRWs exist for $k \geq 0$, both SRWs and Love waves do not exist; Region C: FRWs and Love waves exist for $k \geq 0$, SRWs do not exist; Region D: FRWs and Love waves exist for $k \geq 0$, SRWs exist for $k > k_2(v_L)$.

Chadwick and Jarvis (1979) studied the instability of an elastic half-space by setting the surface wave speed to be zero. Similarly, the dispersion Eq. (51) of Rayleigh waves is reduced to the bifurcation equation by taking $v = 0$. The bifurcation curves of the rubber half-space under biasing field coated with soft or stiff film are plotted in Fig. 8, where its horizontal axis is the dimensionless wavenumber $\tilde{k}$ and the vertical axis is the in-plane stretch λ_1. In Fig. 8, the solid line corresponds to the soft coating film, dashed line to the stiff coating film, and the dot-dashed line to the uncoated elastic half-space. It is observed that the solid line is tangential to the horizontal dot-dashed line $\lambda_1 = \lambda_L$ at point (0, 0.67). Comparison between the two bifurcation curves (dashed and solid lines) in Fig. 8 indicates that, for a given biasing field λ_{B1} satisfying $\lambda_L < \lambda_{B1} < 1$, the limiting instable wavenumber corresponding to a soft film is larger than that to a stiff film. We take the biasing field $\lambda_1 = 0.8$ as an example. The horizontal line $\lambda_1 = 0.8$ in Fig. 8 intercepts the dashed line at point K (0.0629, 0.8) and the solid line at point L (0.0629, 0.8). For the regions on the left-hand side of point K (L) and located above the bifurcation curves, the surface waves with wavenumber $\tilde{k} < 0.0629$ ($\tilde{k} < 6.295$) can propagate stably in the elastic half-space coated with a stiff (soft) film. For the regions on the right-hand side of point K (L) and located below the bifurcation curves, the surface waves in the coated half-space are instable. This feature is consistent with the property of the FRW dispersion curves in Fig. 4.

It should be noted that the dispersion Eq. (54) of Love waves cannot be reduced to a bifurcation equation. If we assume $v = 0$, then Eq. (54) can be written as

$$\mu\lambda_1^{-1} + hk\lambda_2^{-1}\mu^F = 0. \tag{74}$$

The left-hand side of Eq. (74) is positive, meaning that there is no surface buckling mode as Love waves in a coated elastic half-space.

Since the effect of the film on the half-space has been modeled as the effective boundary conditions in which the thickness effect has been dropped, the coating film will always be instable under a compressive biasing field, as already pointed out by Ogden and Steigmann (1997). In other words, when $\tilde{k} \to \infty$, the limiting biasing field λ_1 approaches 1, as shown in Fig. 8. Therefore, the aforementioned instability actually corresponds to a specific wavenumber (or wave mode) and belongs to the catalogue of diffusive buckling. In practice, the thickness effect must be considered to predict the onset of the instability of a coated elastic half-space (Ogden and Steigmann, 1997).

6. Conclusions

The effective boundary conditions, which model the effect of a thin surface layer on the underlying half-space, are derived and employed. By using the Stroh formalism and Barnett-Lothe theory the equations governing surface waves in the coated half-space are obtained. By virtue of the properties of the surface impedance matrix, general criteria are established for the existence and mode multiplicity of surface waves. Under the assumption of uniform pre-deformation, the coated half-space made of restricted isotropic Hadamard material is exemplified and the conditions for existence and the wavenumber ranges of FRWs, SRWs and Love waves are obtained explicitly. Through theoretical analysis, it is found that both the coating film and the pre-deformation can significantly affect the propagation of all kinds of surface waves mentioned above. Therefore, it is possible to tune the wavenumber range and even the existence of each surface wave mode by varying the coating parameter γ and/or the pre-deformation λ_1 for the optimal design of film/substrate structures, which are commonly used as one of the main configurations of wave devices.

Based on our studies, the following conclusions can be drawn. **1)** The existence condition for Love waves is $\gamma > 1$. **2)** The smaller the wavenumber is, the more sensitive Love waves are to the variation of the biasing field. **3)** Distinguishing from Rayleigh waves, the velocity of Love waves increases linearly with pre-stretch λ_1 for the given frequency, a unique feature which could be very useful for defecting internal defects or self-sensing the acoustic devices under biasing fields. **4)** For SRWs, our theoretical analysis indicates that the necessary existence condition of SRWs is that the coating parameter should satisfy $\gamma > \lambda_1^{-2}(\lambda_1^2 - \vartheta_2\lambda_2^2)$. **5)** Our numerical examples show that there are three surface wave modes in an elastic half-space coated with a soft film C_A while there is only one surface wave mode if the coating is a stiff filmC_B. The existence and wavenumber range of each wave mode from our numerical examples are consistent with our theoretical predictions. **6)** The two dispersion curves of FRWs and SRWs in an elastic half-space coated with a soft film C_A will be pushed away from each other by exerting an equal-biaxial compressive biasing field and will be pulled close to each other by an equal-biaxial tensile one. **7)** Mode switching may be observed if the dispersion curves of the two wave modes are very close to each other. **8)** The stability of a coated half-space is also considered using the dispersion equation of Rayleigh waves by setting the wave speed $v = 0$. **9)** A phase diagram is constructed in the (λ_1,γ)-plane to clearly show the existence and mode multiplicity of surface waves, which can be used as guidance on designing actively tunable acoustic devices. In particular, for a specified surface film, the propagation of SRWs may be switched on or off by appropriately tensing the coated half-space.

Finally, it is noted that although the direct thickness effect of the coating film is ignored, the analysis of SRWs is still conducive to better understanding the higher-order Rayleigh waves in an elastic half-space coated with surface layer, and can further offer theoretical guidance for the design of acoustic wave devices based on SRWs.

Acknowledgments

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Appendix A

Ting (2007) obtained the effective boundary conditions of an anisotropic elastic half-space perfectly bonded with an elastic layer. As a direct extension of his treatment to the case with finite pre-deformation, the effective boundary conditions of a deformed elastic half-space overlain by a thin film are obtained in this appendix. Let an undeformed thin anisotropic elastic film be bonded to an undeformed elastic half-space. The film and substrate can be made of different materials. A right-handed Cartesian coordinate system (X_1,X_2,X_3) is attached to the undeformed coated half-space such that the film occupies the region $-H \le X_2 < 0$ while the half-space occupies the domain $X_2 > 0$, see Fig. A1(a). It is assumed that the film and half-space are under uniform pre-deformation with the deformed film and half-space being perfectly bonded together. Another right-handed Cartesian coordinate system (x_1,x_2,x_3) is attached to the deformed coated half-space such that the deformed film occupies the region $-h \le x_2 < 0$ while the half-space occupies the domain $x_2 > 0$, see Fig. A1(b).

From the strong ellipticity condition (3), we conclude that the matrix $\mathbf{C}_2$ in Eq. (7) is positive definite, so that $(7)_2$ can be solved for $\mathbf{u}_{,2}$ as

$$\mathbf{u}_{,2} = \mathbf{D}_0\tau_2 - \mathbf{D}_1^{\mathrm{T}}\mathbf{u}_{,1} - \mathbf{D}_3^{\mathrm{T}}\mathbf{u}_{,3} \tag{A.1}$$

in which

$$\mathbf{D}_0 = \mathbf{C}_2^{-1}, \quad \mathbf{D}_1 = \mathbf{C}_1^{\mathrm{T}}\mathbf{D}_0, \quad \mathbf{D}_3 = \mathbf{C}_3^{\mathrm{T}}\mathbf{D}_0 \tag{A.2}$$

Substitution of Eq. (A.1) into Eqs. $(7)_1$ and $(7)_3$ yields

$$\begin{aligned}\tau_1 &= \hat{\mathbf{E}}_0\tau_2 + \hat{\mathbf{E}}_1\mathbf{u}_{,1} + \hat{\mathbf{E}}_3\mathbf{u}_{,3}\\ \tau_3 &= \breve{\mathbf{E}}_0\tau_2 + \breve{\mathbf{E}}_1\mathbf{u}_{,1} + \breve{\mathbf{E}}_3\mathbf{u}_{,3}\end{aligned} \tag{A.3}$$

where

$$\begin{aligned}\hat{\mathbf{E}}_0 &= \hat{\mathbf{C}}_2\mathbf{D}_0, \quad \hat{\mathbf{E}}_1 = \hat{\mathbf{C}}_1 - \hat{\mathbf{E}}_0\mathbf{C}_1, \quad \hat{\mathbf{E}}_3 = \hat{\mathbf{C}}_3 - \hat{\mathbf{E}}_0\mathbf{C}_3\\ \breve{\mathbf{E}}_0 &= \breve{\mathbf{C}}_2\mathbf{D}_0, \quad \breve{\mathbf{E}}_1 = \breve{\mathbf{C}}_1 - \breve{\mathbf{E}}_0\mathbf{C}_1, \quad \breve{\mathbf{E}}_3 = \breve{\mathbf{C}}_3 - \breve{\mathbf{E}}_0\mathbf{C}_3\end{aligned} \tag{A.4}$$

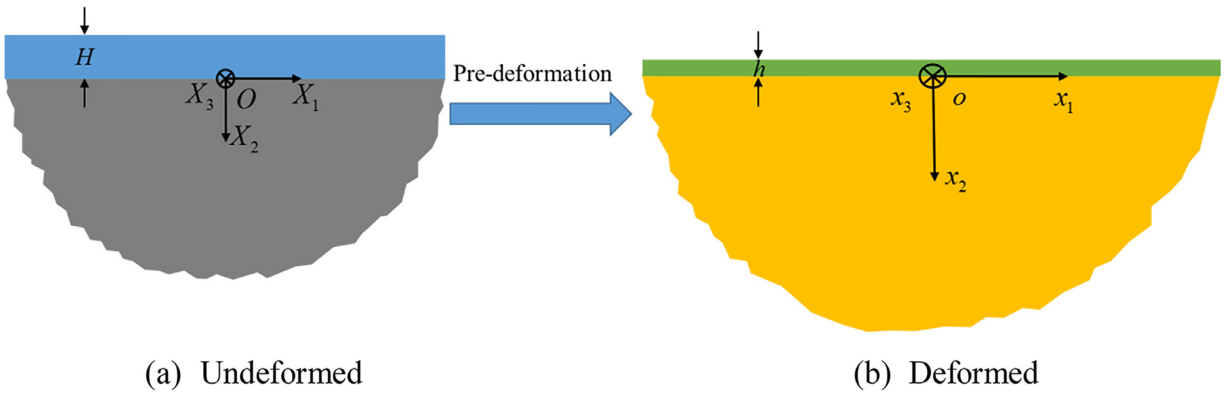

Fig. A.1. The coated half-space in (undeformed) reference and (deformed) initial configurations.

By substituting Eq. (A.3) into the equations of motion (6) and assuming the effective elastic coefficients be independent of x_1 and x_3, we obtain

$$\tau_{2,2} + \mathbf{D}_1 \tau_{2,1} + \mathbf{D}_3 \tau_{2,3} + \mathbf{G}_1 \mathbf{u}_{,11} + \mathbf{G}_2 \mathbf{u}_{,13} + \mathbf{G}_3 \mathbf{u}_{,33} = \rho \mathbf{u}_{,tt} \tag{A.5}$$

where

$$\mathbf{D}_1 = \hat{\mathbf{E}}_0, \quad \mathbf{D}_3 = \breve{\mathbf{E}}_0, \quad \mathbf{G}_1 = \hat{\mathbf{E}}_1, \quad \mathbf{G}_2 = \hat{\mathbf{E}}_3 + \breve{\mathbf{E}}_1, \quad \mathbf{G}_3 = \breve{\mathbf{E}}_3 \tag{A.6}$$

Since $\hat{\mathbf{C}}_2 = \mathbf{C}_1^{\mathrm{T}}$ and $\breve{\mathbf{C}}_2 = \mathbf{C}_3^{\mathrm{T}}$, then $\mathbf{D}_1$ and $\mathbf{D}_3$ defined in Eqs. (A.2) and (A.6) can be shown to be equivalent.

Equations (A.1)-(A.6) are appropriate for an elastic body of arbitrary shape. For an elastic film, we introduce the Taylor expansion of the transverse stress vector as

$$\begin{aligned} \tau_2^F|_{x_2=-h} &= \tau_2^F|_{x_2=0^-} - h\frac{\partial \tau_2^F}{\partial x_2}|_{x_2=0^-} \\ &+ \ldots + (-1)^n \frac{h^n}{n!}\frac{\partial^n \tau_2^F}{\partial^n x_2}|_{x_2=0^-} + o(h^n) \end{aligned} \tag{A.7}$$

where the superscript F denotes physical quantities/parameters in the film, h is the thickness of the deformed film with $x_2 = 0^-$ being its bottom surface. The upper surface of the film is assumed to be tractions-free, i.e.

$$\tau_2^F|_{x_2=-h} = \mathbf{0} \tag{A.8}$$

For a thin film, h is small. Then, in the first-order approximation, Eq. (A.7) can be simplified to

$$\tau_2^F = h\frac{\partial \tau_2^F}{\partial x_2} \quad (x_2 = 0^-) \tag{A.9}$$

Combination of Eqs. (A.5) and (A.9) yields

$$\begin{aligned} &\frac{1}{h}\tau_2^F + \mathbf{D}_1^F \tau_{2,1}^F + \mathbf{D}_3^F \tau_{2,3}^F + \mathbf{G}_1^F \mathbf{u}_{,11}^F + \mathbf{G}_2^F \mathbf{u}_{,13}^F \\ &+ \mathbf{G}_3^F \mathbf{u}_{,33}^F = \rho^F \mathbf{u}_{,tt}^F \quad (x_2 = 0^-) \end{aligned} \tag{A.10}$$

The assumption that the deformed film is perfectly bonded to the half-space means that $\boldsymbol{\tau}_2$ and $\mathbf{u}$ are continuous across the interface $x_2 = 0$. Then Eq. (A.10) can be rewritten as

$$\frac{\tau_2}{h} + \mathbf{D}_1^F \tau_{2,1} + \mathbf{D}_3^F \tau_{2,3} +! \mathbf{G}_1^F \mathbf{u}_{,11} + \mathbf{G}_2^F \mathbf{u}_{,13} + \mathbf{G}_3^F \mathbf{u}_{,33} = \rho^F \mathbf{u}_{,tt} \quad (x_2 = 0) \tag{A.11}$$

The analysis by Benveniste (2006) indicates that by ignoring the terms $\mathbf{D}_1^F \tau_{2,1}$ and $\mathbf{D}_3^F \tau_{2,3}$ in Eq. (A.11), the direct thickness effect of the coating film is ignored, while the effects of elasticity and inertia are included. Thus, we obtain the effective boundary conditions of a deformed half-space overlain by a deformed film without the direct thickness effect as

$$\tau_2 + h\mathbf{G}_1^F \mathbf{u}_{,11} + h\mathbf{G}_2^F \mathbf{u}_{,13} + h\mathbf{G}_3^F \mathbf{u}_{,33} = h\rho^F \mathbf{u}_{,tt} \tag{A.12}$$

The derivation of the effective boundary conditions is similar to the treatment in Mindlin (1963) and Tiersten (1969). It can be proved that the effective boundary conditions (A.12) are consistent with the surface elasticity theory proposed by Gurtin and Murdoch (Gurtin and Murdoch, 1975). In the main text of the present paper, surface waves in a deformed elastic half-space overlain by a coating film are investigated based on Eq. (A.12), which approximates the effect of the film in terms of the generalized boundary conditions.

During the derivation of effective boundary conditions, we have assumed that the reference configuration is similar to the initial configuration in terms of the geometry and that the effective elastic tensor $\mathbf{A}_0^F$ is independent of the in-plane coordinates x_1 and x_3. For an isotropic material, one possible biasing field meeting the above assumption is that the thin film and the half-space are equi-biaxially tensed/compressed with primary stretches $(\lambda_1, \lambda_2, \lambda_1)$ applied to the substrate and $(\lambda_1, \lambda_2^F, \lambda_1)$ to the coating film, while keeping tractions-free on the plane $X_2 = \text{const}$. Then the deformation gradient tensors of the substrate and the film are respectively

$$\mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}, \quad \mathbf{F}^F = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2^F & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} \tag{A.13}$$

If the strain energy functions of the film and half-space are known as $\Omega^F = \Omega^F(\mathbf{F}^F)$ and $\Omega = \Omega(\mathbf{F})$, the effective elastic coefficients corresponding to the film and half-space can be obtained from Eq. (2). For the restricted isotropic Hadamard material specified by Eqs. (35) and (36), the tractions-free conditions on the plane of $X_2 = \text{const.}$ give unique solutions of λ_2^F and λ_2, the two being equal in the case considered in the present paper. We shall refer to the case that the deformation gradient tensor in both the substrate and the film are identical as the uniform pre-deformation state of the coated half-space.

Appendix B

In the absence of pre-deformation, Murdoch (1976) investigated the propagation of surface waves in an elastic half-space with material boundary based on the surface elasticity proposed by Gurtin and Murdoch (1975). He found that if the surface elasticity satisfies

$$\mu_0/\rho_0 < \mu/\rho \tag{B.1}$$

then Love waves can propagate in the half-space, with the dispersion equation being

$$l^2 k^2 = (1 - s^2)/(s^2 - s_0^2)^2 \tag{B.2}$$

where $l = \rho_0/\rho$, $s_0 = v_0/v_b$, $s = v/v_b$, $v_0 = \sqrt{\mu_0/\rho_0}$, $v_b = \sqrt{\mu/\rho}$, ρ_0 and μ_0 are respectively the surface density and surface shear modulus defined in Murdoch (1976). On the other hand, by setting $\mu_0 = h\mu^F$ and $\rho_0 = h\rho^F$, our dispersion Eq. (62) of Love waves degenerates to

$$k = \frac{\mu\sqrt{1-\rho v^2/\mu}}{h\rho^F v^2 - h\mu^F} = \frac{v_b^2\sqrt{1-v^2/v_b^2}}{l(v^2-v_0^2)} = \frac{\sqrt{1-s^2}}{l(s^2-s_0^2)} \tag{B.3}$$

During the derivation of Eq. (B.3), the biasing field is dropped. It is obvious that Eq. (B.3) is equivalent to Eq. (B.2).

Murdoch (1976) also obtained the dispersion equation of Rayleigh waves in an elastic half-space with material boundary. When the residual stress is absent, his dispersion equation becomes

$$CD(1-ST)k^2 + s^2(CT+DS)k + [4ST-(2-s^2)^2] = 0 \tag{B.4}$$

where

$$\begin{aligned} &C = -\frac{lv^2}{v_b^2}, \quad D = \frac{\lambda_0 + 2\mu_0}{\rho v_b^2} + C \\ &S = \sqrt{1-s^2}, \quad T = \sqrt{1-s^2/s_p^2} \\ &s_p = v_p/v_b, \quad v_p = \sqrt{(\lambda+2\mu)/\rho} \end{aligned} \tag{B.5}$$

In Eq. (B.5), λ_0 is another surface Lamé constant (Murdoch, 1976). The dispersion Eq. (50) of Rayleigh waves in the present paper is equal to Eq. (B.4) by appropriately choosing the coating parameters. This is presented below.

The elastic constants of the undeformed isotropic coating film and substrate are respectively

$$\begin{aligned} A^F_{0ijkl} &= \lambda^F\delta_{ij}\delta_{kl} + \mu^F(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \\ A_{0ijkl} &= \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \end{aligned} \tag{B.6}$$

Substitution of Eq. (B.6) into Eq. (51) yields

$$\begin{aligned} -\Gamma^F\rho^F v^2h^2k^2 + \left(-\rho^F v^2\frac{S^2-1}{ST-1}T\mu + \Gamma^F\frac{S^2-1}{ST-1}S\mu\right)hk \\ + \frac{(S^2-1)^2ST-(S^2-2ST+1)^2}{(ST-1)^2}\mu^2 = 0 \end{aligned} \tag{B.7}$$

in which

$$\Gamma^F = \frac{2\mu^F}{\lambda^F+2\mu^F}\lambda^F + 2\mu^F - \rho^F v^2. \tag{B.8}$$

Eq. (B.7) can be rewritten as

$$\begin{aligned} &-\frac{\rho^F}{\mu}v^2\frac{\Gamma^F}{\mu}(1-ST)h^2k^2 + s^2\left(-\frac{\rho^F}{\mu}v^2T + \frac{\Gamma^F}{\mu}S\right)hk \\ &+[4ST-(2-s^2)^2] = 0 \end{aligned} \tag{B.9}$$

Comparison between Eqs. (B.4) and (B.9) indicates that by setting

$$\lambda_0 = \frac{2h\mu^F\lambda^F}{\lambda^F+2\mu^F}, \quad \mu_0 = h\mu^F, \quad \rho_0 = h\rho^F \tag{B.10}$$

the two equations are then identical to each other. It is noted that Murdoch (1976) also made a comparison of the surface elasticity theory (Gurtin and Murdoch, 1975) with the Mindlin-Tiersten effective boundary conditions (Mindlin, 1963; Tiersten, 1969) and derived a consistency condition, which is equal to Eq. (B.10). Chen et al., (2014) obtained a similar condition when investigating the elasticity for a cylindrical surface.

Appendix C

In this appendix, the conditions for existence of FRWs and SRWs and the associated wavenumber ranges in a deformed coated half-space made of restricted isotropic Hadamard material are obtained based on Theorems 1 and 2 in the main text.

Table C1
Six different cases in Group A.

Case	Relation between λ_1 and λ_2	Relation between $N^0_{311}(0)$ and $N^0_{322}(0)$
A1	$\lambda_1 > \lambda_2$	$N^0_{311}(0) \le N^0_{322}(0) < 0$
A2	$\lambda_1 = \lambda_2$	$N^0_{322}(0) = 0, N^0_{311}(0) \le 0$
A3	$\lambda_2^2(2\lambda_2^2-\theta_2^2)/\theta_2^2 \le \lambda_1^2 < \lambda_2^2$	$N^0_{311}(0) \le -N^0_{322}(0) < 0$
A4	$\vartheta_2\lambda_2^2 < \lambda_1^2 < \lambda_2^2(2\lambda_2^2-\theta_2^2)/\theta_2^2$	$N^0_{322}(0) > 0, N^0_{311}(0) = 0$
A5	$\lambda_1^2 = \vartheta_2\lambda_2^2 \quad (\vartheta_2 \ne 1)$	$N^0_{322}(0) > 0, N^0_{311}(0) = 0$
A6	$\lambda_1^2 < \vartheta_2\lambda_2^2$	$N^0_{322}(0) \ge N^0_{311}(0) > 0$

1. Existence of FRWs

1.1. Solutions of $y_1(0,k) > 0$

Inequality $y_1(0,k) > 0$ is satisfied if both the inequalities

$$\begin{aligned} &M_{11}(0) + M_{22}(0) - k[N^0_{311}(0) + N^0_{322}(0)] > 0 \\ &\text{and } H_1(0)k^2 + H_2(0)k + H_3(0) > 0 \end{aligned} \tag{C.1}$$

hold simultaneously. In Eq. (C.1), $H_1(v)$, $H_2(v)$ and $H_3(v)$ are defined in Eq. (52). From inequality (47) we obtain $N^0_{311}(0) \le N^0_{322}(0)$. For all the situations considered, we can obtain $\alpha(0) \ge \beta(0) \ge 1$ or $\alpha(0) \le \beta(0) < 1$, from which we are able to arrive at

$$M_{11}(0) \ge 0, \quad M_{22}(0) \ge 0 \tag{C.2}$$

The equality in Eq. (C.2) may hold when $\lambda_1 = \lambda_2$. Let us analyze the limit

$$\begin{aligned} \lim_{\alpha(0),\beta(0)\to 1} M_{11}(0) &= \lim_{\alpha(0),\beta(0)\to 1} M_{22}(0) \\ &= \lim_{\alpha(0),\beta(0)\to 1}\left[\frac{\alpha^2(0)-1}{\alpha(0)\beta(0)-1}\mu J^{-1}\lambda_2^2\right] \end{aligned} \tag{C.3}$$

By making use of the L'Hôpital's rule, it can be proved that for all the cases we have $[\alpha^2(0)-1]/[\alpha(0)\beta(0)-1] > 0$. Thus, instead of Eq. (C.2), we have

$$M_{11}(0) > 0, \quad M_{22}(0) > 0 \tag{C.4}$$

In the following, the discussion on inequality (C.1) will be divided into two groups, i.e. Group A

$$H_3(0) > 0 \tag{C.5}$$

and Group B

$$H_3(0) \le 0 \tag{C.6}$$

1.1.1. Group A. We first consider Group A. The discriminant of inequality $(C.1)_2$ is $\Delta(0)$ with $\Delta(v)$ being defined in Eq. $(57)_2$. $\Delta(0) \ge 0$ is true for all the cases in Group A, which is proved below. If $N^0_{311}(0)N^0_{322}(0) < 0$, it is obvious that $\Delta(0) > 0$ due to inequality (C.5); If $N^0_{311}(0)N^0_{322}(0) \ge 0$, $\Delta(0)$ can be rewritten as

$$\begin{aligned} \Delta(0) = &[N^0_{322}(0)M_{11}(0) - N^0_{311}(0)M_{22}(0)]^2 \\ &-4N^0_{311}(0)N^0_{322}(0)M^2_{12}(0) \end{aligned} \tag{C.7}$$

Since $M_{12}(0)$ is purely imaginary or zero, we obtain $M^2_{12}(0) \le 0$, based on which we have $\Delta(0) \ge 0$ from Eq. (C.7). As a conclusion, we find $\Delta(0) \ge 0$ is valid for all the cases in Group A.

Furthermore, from Eq. (42) we can prove

$$\vartheta_2 \le (2\lambda_2^2 - \theta_2^2)/\theta_2^2 \le 1 \tag{C.8}$$

where θ_2 is defined in Eq. $(41)_3$ and ϑ_2 is defined in Eq. (46). The equality in Eq. (C.8) holds if and only if $f''(J) = 0$. Based on inequality (C.8) we further divide Group A into six different cases, which are listed in Table C1.

In Case A1, $N^0_{311}(0) \le N^0_{322}(0) < 0$. Thus, inequality $(C.1)_1$ is satisfied automatically. The solution of inequality $(C.1)_2$ is

$$k < k_1(0) \quad \cup \quad k > k_2(0) \tag{C.9}$$

where $k_1(v)$ and $k_2(v)$ are defined in Eqs. $(57)_3$ and $(57)_4$ and the symbol $\cup$ denotes the union of sets. In view of $N^0_{311}(0) \le N^0_{322}(0) <$

0 and inequalities (C.4) and (C.5), we obtain $k_1(0) < 0$ and $k_2(0) < 0$ in this case. Therefore the solution of inequality (C.1) is $k \geq 0$ in Case A1.

In Case A2, $N^0_{322}(0) = 0$ and $N^0_{311}(0) \leq 0$, then (C.1) degenerates to

$$M_{11}(0) + M_{22}(0) - kN^0_{311}(0) > 0 \quad \text{and} \quad -N^0_{311}(0)M_{22}(0)k + H_3(0) > 0 \tag{C.10}$$

If $f'(J)=0$ we have $N^0_{322}(0) = 0$ and $N^0_{311}(0) = 0$. As a result, inequality (C.10) automatically holds and its solution is $k \geq 0$. If $f'(J) > 0$, we have $N^0_{311}(0) < 0$. Inequalities $(C.10)_1$ and $(C.10)_2$ are satisfied automatically due to inequalities (C.4) and (C.5). Consequently, the solution of (C.1) in Case A2 is $k \geq 0$.

In Case A3, we have $N^0_{311}(0) \leq -N^0_{322}(0) < 0$ and inequality $(C.1)_1$ is satisfied automatically. The solution of inequality $(C.1)_2$ is

$$k_2(0) < k < k_1(0) \tag{C.11}$$

By using inequalities (C.4), (C.5) and $N^0_{311}(0) \leq -N^0_{322}(0) < 0$ we can prove $k_1(0) > 0$ and $k_2(0) < 0$. Thus the solution of inequality (C.1) is $0 \leq k < k_1(0)$ in Case A3.

In Case A4, we have $-N^0_{322}(0) < N^0_{311}(0) < 0$. The solution of inequality (C.1) is

$$k < k_{p0} \ \cap \ k_2(0) < k < k_1(0) \tag{C.12}$$

where

$$k_{p0} = \frac{M_{11}(0) + M_{22}(0)}{N^0_{311}(0) + N^0_{322}(0)} \tag{C.13}$$

By using inequalities $-N^0_{322}(0) < N^0_{311}(0) < 0$ and (C.4), we can prove $k_{p0} > 0$. In addition, we obtain $k_1(0) > 0$ and $k_2(0) < 0$ with the help of inequalities (C.4) and (C.5). Let us consider

$$k_1(0) - k_{p0} = \frac{[N^0_{311}(0) - N^0_{322}(0)][N^0_{311}(0)M_{22}(0) - N^0_{322}(0)M_{11}(0)] - [N^0_{311}(0) + N^0_{322}(0)]\sqrt{\Delta(0)}}{2N^0_{311}(0)N^0_{322}(0)[N^0_{311}(0) + N^0_{322}(0)]} \tag{C.14}$$

The inequalities $N^0_{311}(0) < 0$, $N^0_{322}(0) > 0$ and $N^0_{311}(0) + N^0_{322}(0) > 0$ lead to

$$N^0_{311}(0)N^0_{322}(0)[N^0_{311}(0) + N^0_{322}(0)] < 0 \tag{C.15}$$

We can further prove that

$$[N^0_{311}(0) - N^0_{322}(0)][N^0_{311}(0)M_{22}(0) - N^0_{322}(0)M_{11}(0)] > 0 \quad [N^0_{311}(0) + N^0_{322}(0)]\sqrt{\Delta(0)} > 0 \tag{C.16}$$

Therefore, whether $k_1(0) - k_{p0}$ as given in Eq. (C.14) is positive or negative depends only on the comparison of values of

$$[N^0_{311}(0) - N^0_{322}(0)][N^0_{311}(0)M_{22}(0) - N^0_{322}(0)M_{11}(0)] \tag{C.17}$$

and

$$[N^0_{311}(0) + N^0_{322}(0)]\sqrt{\Delta(0)}. \tag{C.18}$$

Due to inequality (C.5), the absolute value of $N^0_{311}(0) - N^0_{322}(0)$ is larger than that of $N^0_{311}(0) + N^0_{322}(0)$, and the absolute value of $N^0_{322}(0)M_{11}(0) - N^0_{311}(0)M_{22}(0)$ is larger than that of $\sqrt{\Delta(0)}$. Therefore we conclude that

$$[N^0_{311}(0) - N^0_{322}(0)][N^0_{311}(0)M_{22}(0) - N^0_{322}(0)M_{11}(0)] > [N^0_{311}(0) + N^0_{322}(0)]\sqrt{\Delta(0)} \tag{C.19}$$

As a result, we obtain $k_1(0) < k_{p0}$. Thus, the solution of inequality (C.1) in Case A4 is $0 \leq k < k_1(0)$.

In Case A5, we know $N^0_{322}(0) > 0$ and $N^0_{311}(0) = 0$. Inequality (C.1) degenerates to

$$M_{11}(0) + M_{22}(0) - kN^0_{322}(0) > 0 \quad \text{and} \quad -N^0_{322}(0)M_{11}(0)k + H_3(0) > 0 \tag{C.20}$$

with the following solution

$$k < k_{p0} \ \cap \ k < k_{p1}, \tag{C.21}$$

where $\cap$ represents intersection of sets and

$$k_{p1} = \frac{H_3(0)}{N^0_{322}(0)M_{11}(0)} \tag{C.22}$$

From inequalities (C.4) and (C.5) we have

$$k_{p1} > 0 \ \text{and} \ k_{p0} > 0 \tag{C.23}$$

Let us consider

$$k_{p1} - k_{p0} = \frac{M^2_{12}(0)}{N^0_{322}(0)M_{11}(0)} - \frac{M_{11}(0)}{N^0_{322}(0)} \tag{C.24}$$

We immediately know that $k_{p1} - k_{p0} < 0$ in view of inequalities (C.4). Thus, the solution of inequality (C.20) is $0 \leq k < k_{p1}$. We can further prove that

$$\lim_{\lambda_1^2 \to \vartheta\lambda_2^2} k_1(0) = k_{p1} \tag{C.25}$$

Thus, the solution of inequality (C.1) in Cases A4 and A5 can be uniformly written as $0 \leq k < k_1(0)$.

In Case A6, we know $N^0_{322}(0) \geq N^0_{311}(0) > 0$. The solution of inequality $(C.1)_1$ is $k < k_{p0}$ and the solution of inequality $(C.1)_2$ is given in Eq. (C.9). By using inequalities $M_{11}(0) \geq M_{22}(0)$, (C.4), (C.5) and $N^0_{322}(0) \geq N^0_{311}(0) > 0$, we obtain

$$0 < k_1(0) \leq \frac{M_{22}(0)}{N^0_{322}(0)} \leq \frac{M_{11}(0) + M_{22}(0)}{N^0_{311}(0) + N^0_{322}(0)} = k_{p0} \tag{C.26}$$

and

$$k_2(0) \geq \frac{M_{11}(0)}{N^0_{311}(0)} \geq \frac{M_{22}(0) + M_{11}(0)}{N^0_{311}(0) + N^0_{322}(0)} = k_{p0} \tag{C.27}$$

Thus, the solution of inequality (C.1) is $0 \leq k < k_1(0)$.

The discussions from Case A1 to Case A6 show that the solution of inequality (C.1) is $k \geq 0$ if the biasing field satisfies $\lambda_1 \geq \lambda_2$, and the solution of inequality (C.1) is $0 \leq k < k_1(0)$ if the biasing field satisfies $\lambda_1 < \lambda_2$.

1.1.2. Group B. We now consider Group B. First we will prove that the inequality (C.6) implies the relation $\lambda_1 \leq \lambda_2$. Actually, if $\lambda_1 > \lambda_2$, from the expressions of α and β in Eq. (41) and the relation (42) we obtain

$$\alpha(0) \geq \beta(0) > 1 \tag{C.28}$$

$H_3(0)$ can be expressed as

$$H_3(0) = \frac{\mu^2 J^{-2}\lambda_2^4}{[\alpha(0)\beta(0) - 1]^2}\{[1 - \alpha^2(0)]^2\alpha(0)\beta(0) - [\alpha^2(0) - 2\alpha(0)\beta(0) + 1]^2\} \tag{C.29}$$

Owing to

$$\begin{aligned}&[1 - \alpha^2(0)]^2\alpha(0)\beta(0) - [\alpha^2(0) - 2\alpha(0)\beta(0) + 1]^2 \\ &> [1 - \alpha^2(0)]^2 - [\alpha^2(0) - 2\alpha(0)\beta(0) + 1]^2 \\ &= 4\alpha(0)[\alpha(0)\beta(0) - 1][\alpha(0) - \beta(0)] \geq 0\end{aligned} \tag{C.30}$$

and inequality (C.28) we obtain $H_3(0) > 0$. Consequently, the assumption $H_3(0) \leq 0$ means $\lambda_1 \leq \lambda_2$ and as a result, $N^0_{322}(0) \geq 0$. In addition, we obtain $M_{11}(\nu) \geq M_{22}(\nu)$, as a result of $\lambda_1 \leq \lambda_2$. The equation $\Delta(0)=0$ can be regarded as a quadratic equation of $N^0_{311}(0)$ and its two solutions, denoted by Λ_1 and Λ_2, are

$$\Lambda_1 = \frac{-\left[\sqrt{-H_3(0)} + \sqrt{-M^2_{12}(0)}\right]^2 N^0_{322}(0)}{M^2_{22}(0)}$$

$$\Lambda_2 = \frac{-\left[\sqrt{-H_3(0)} - \sqrt{-M^2_{12}(0)}\right]^2 N^0_{322}(0)}{M^2_{22}(0)} \tag{C.31}$$

Table C2
Six different cases in Group B.

Case	Ranges of $N^0_{311}(0)$ and $N^0_{322}(0)$	Ranges of $\Delta(0)$
B1	$N^0_{322}(0) = 0$ and $N^0_{311}(0) \le 0$	$\Delta(0) \ge 0$
B2	$N^0_{322}(0) > 0$ and $N^0_{311}(0) < \Lambda_1$	$\Delta(0) > 0$
B3	$N^0_{322}(0) > 0$ and $\Lambda_1 \le N^0_{311}(0) \le \Lambda_2$	$\Delta(0) \le 0$
B4	$N^0_{322}(0) > 0$ and $\Lambda_2 < N^0_{311}(0) < 0$	$\Delta(0) > 0$
B5	$N^0_{322}(0) > 0$ and $N^0_{311}(0) = 0$	$\Delta(0) > 0$
B6	$N^0_{322}(0) > 0$ and $N^0_{311}(0) > 0$	$\Delta(0) > 0$

in which $\Lambda_1 \le \Lambda_2 \le 0$. Thus, Group B also can be further divided into six cases, which are listed in Table C2.

Due to the inequality

$$\Lambda_1 + \frac{M_{11}(0)}{M_{22}(0)} N^0_{322}(0) = \frac{2\left[H_3(0) - \sqrt{-H_3(0)}\sqrt{-M^2_{12}(0)}\right]}{M^2_{22}(0)} N^0_{322}(0) \le 0 \tag{C.32}$$

we obtain

$$\Lambda_1 \le -\frac{M_{11}(0)}{M_{22}(0)} N^0_{322}(0) \tag{C.33}$$

The equality in the above formula holds if and only if $N^0_{322}(0) = 0$ or $H_3(0) = 0$.

In Case B1, we have $N^0_{322}(0) = 0$ and $N^0_{311}(0) \le 0$. If $f''(J) = 0$ we have $N^0_{311}(0) = 0$ and inequality $(C.1)_2$ has no solution, which means inequality (C.1) has no solution, i.e. $k \in \emptyset$. If $f''(J) > 0$, we have $N^0_{311}(0) < 0$ and the solution of inequality (C.1) is $k > H_3(0)/[N^0_{311}(0)M_{22}(0)]$. The limit

$$\lim_{N^0_{311}(0) \to 0^-} H_3(0)/\left[N^0_{311}(0)M_{22}(0)\right] = +\infty \tag{C.34}$$

implies that the solution of inequality (C.1) can be written uniformly as $k > H_3(0)/[N^0_{311}(0)M_{22}(0)]$.

In Case B2, with the help of inequality (C.32) we obtain $N^0_{311}(0) < -N^0_{322}(0)M_{11}(0)/M_{22}(0) \le -N^0_{322}(0)$. Inequality $(C.1)_1$ is satisfied automatically and the solution of inequality $(C.1)_2$ is given by formula (C.11). In addition, we can prove that $k_1(0) > 0$ and $k_2(0) \ge 0$. Therefore the solution of inequality (C.1) is $k_2(0) < k < k_1(0)$ in Case B2.

In Case B3, $\Delta(0)$ is negative. Therefore inequality $(C.1)_2$ has no solution, and as a result, inequality (C.1) has no solution, i.e. $k \in \emptyset$.

In Case B4, we obtain $-M_{11}(0)N^0_{322}(0)/M_{22}(0) < N^0_{311}(0) < 0$ from which we find that inequality $(C.1)_2$ cannot be satisfied, and therefore inequality (C.1) has no solution, i.e. $k \in \emptyset$.

In Case B5, inequality (C.1) degenerates to inequality (C.20). Because of inequalities (C.4) and (C.6), inequality (C.20) has no solution, i.e. $k \in \emptyset$.

In Case B6, the solution of inequality $(C.1)_1$ is $k < k_{p0}$ and the solution of inequality $(C.1)_2$ is formula (C.9). It can be proved that $k_1(0) \le 0$ and $k_2(0) > 0$.

Let us consider

$$k_2(0) - k_{p0} = \frac{[N^0_{311}(0) - N^0_{322}(0)][N^0_{311}(0)M_{22}(0) - N^0_{322}(0)M_{11}(0)] + [N^0_{311}(0) + N^0_{322}(0)]\sqrt{\Delta(0)}}{2N^0_{311}(0)N^0_{322}(0)[N^0_{311}(0) + N^0_{322}(0)]} \tag{C.35}$$

Due to the condition of Case B6 as indicated in Table C2 and $M_{11}(0) \ge M_{22}(0)$, we obtain $k_2(0) \ge k_{p0}$. Consequently, inequality (C.1) has no solution, i.e. $k \in \emptyset$.

As a conclusion of the above discussions in Group B, we find that if the biasing field satisfies $N^0_{322}(0) = 0$, the solution of inequality (C.1) is $k > H_3(0)/N^0_{311}(0)M_{22}(0)$; if the biasing field satisfies $N^0_{322}(0) > 0$ and $N^0_{311}(0) < \Lambda_1$, the solution is $k_2(0) < k < k_1(0)$; for the other cases in Group B, inequality (C.1) has no solution.

Table C3
Solutions of $y_1(0,k) > 0$ in all situations.

Conditions	Wave number range
$\lambda_1 > \lambda_2$	$k \ge 0$
$\lambda_1 = \lambda_2$ and $H_3(0) > 0$	$k \ge 0$
$\lambda_1 = \lambda_2$ and $H_3(0) \le 0$	$k > H_3(0)/N^0_{311}(0)M_{22}(0)$
$\lambda_1 < \lambda_2$ and $H_3(0) > 0$	$0 \le k < k_1(0)$
$\lambda_1 < \lambda_2$, $H_3(0) \le 0$ and $N^0_{311}(0) < \Lambda_1$	$k_2(0) < k < k_1(0)$
$\lambda_1 < \lambda_2$, $H_3(0) \le 0$ and $N^0_{311}(0) \ge \Lambda_1$	$k \in \emptyset$

Table C4
Six different cases of $y_i(v_L,k) < 0, \quad (i = 1, 2)$.

Case	Value of γ	Relation between $N^0_{311}(0)$ and $N^0_{322}(0)$
I	$\gamma < 1 - \lambda_1^{-2}\lambda_2^2$	$N^0_{311}(v_L) \le N^0_{322}(v_L) < 0$
II	$\gamma = 1 - \lambda_1^{-2}\lambda_2^2$	$N^0_{311}(v_L) \le 0, \quad N^0_{322}(v_L) = 0$
III	$1 - \lambda_2^2\lambda_1^{-2} < \gamma \le 1 + (\theta_2^2 - 2\lambda_2^2)\lambda_2^2\lambda_1^{-2}\theta_2^{-2}$	$N^0_{311}(v_L) \le -N^0_{322}(v_L) < 0$
IV	$1 + (\theta_2^2 - 2\lambda_2^2)\lambda_2^2\lambda_1^{-2}\theta_2^{-2} < \gamma < 1 - \vartheta_2\lambda_2^2\lambda_1^{-2}$	$-N^0_{322}(v_L) < N^0_{311}(v_L) < 0$
V	$\gamma = 1 - \vartheta_2\lambda_2^2\lambda_1^{-2}$	$N^0_{311}(v_L) = 0, N^0_{322}(v_L) \ge 0$
VI	$\gamma > 1 - \vartheta_2\lambda_2^2\lambda_1^{-2}$	$N^0_{311}(v_L) > 0, N^0_{322}(v_L) > 0$

1.1.3. Groups A and B. The combination of discussions in Groups A and B gives the solutions of $y_1(0,k) > 0$ in various situations, which are summarized in Table C3.

1.2. Solutions of $y_1(v_L,k) < 0$

The inequality $y_1(v_L,k) < 0$ holds if either

$$M_{11}(v_L) - k[N^0_{311}(v_L) + N^0_{322}(v_L)] < 0$$
$$\text{or } N^0_{311}(v_L)N^0_{322}(v_L)k^2 - N^0_{322}(v_L)M_{11}(v_L)k + M^2_{12}(v_L) < 0 \tag{C.36}$$

is satisfied, in which

$$N^0_{311}(v_L) = -h\mu^F\lambda_2^{-1} + h\mu^F J^{-1}\vartheta_2\lambda_2^2 + \lambda_2^{-1}h\mu\rho^F\rho^{-1},$$
$$N^0_{322}(v_L) = -h\mu^F\lambda_2^{-1} + h\mu^F J^{-1}\lambda_2^2 + \lambda_2^{-1}h\mu\rho^F\rho^{-1}. \tag{C.37}$$

The inequality (C.36) is discussed in detail in the following by using the properties

$$N^0_{311}(v_L) \le N^0_{322}(v_L), \quad M_{11}(v_L) \ge 0, \quad M^2_{12}(v_L) < 0 \tag{C.38}$$

where

$$M_{11}(v_L) = \mu J^{-1}\lambda_2^2\beta(v_L) \quad M_{12}(v_L) = \mathrm{i}\mu J^{-1}\lambda_2^2 \tag{C.39}$$

The equality in (C.38) holds if and only if $f''(J) = 0$. The discriminant of inequality $(C.36)_2$ is $\Delta(v_L)$, with $\Delta(v)$ being defined in Eq. $(57)_2$.

The analysis of inequality (C.36) will be divided into six different cases, which are listed in Table C4. The parameters γ and ϑ_2 in Table C4 are, respectively, defined in Eqs. $(57)_1$ and (46). It is noted that Cases III and IV imply the condition $f''(J) > 0$, and therefore when discussing these two cases, we do not need to consider the equalities in formulas (C.38).

In Case I, we have $N^0_{311}(v_L) \le N^0_{322}(v_L) < 0$ and therefore inequality $(C.36)_1$ cannot be satisfied. In this case, $\Delta(v_L) > 0$ and the solution of inequality $(C.36)_2$ is

$$k_1(v_L) < k < k_2(v_L) \tag{C.40}$$

In inequality (C.40), $k_1(v)$ and $k_2(v)$ are, respectively, defined in Eqs. $(57)_3$ and $(57)_4$. By using inequality (C.38) we can prove $k_1(v_L) < 0$ and $k_2(v_L) > 0$. Thus the solution of inequality (C.36) in this case is $0 \le k < k_2(v_L)$.

Table C5
Solutions of $y_1(v_L,k) < 0$ in all situations.

Conditions	Wavenumber range
$\gamma < \lambda_1^{-2}(\lambda_1^2 - \lambda_2^2)$	$0 \le k < k_2(v_L)$
$\gamma \ge \lambda_1^{-2}(\lambda_1^2 - \lambda_2^2)$	$k \ge 0$

In Case II, we obtain $N^0_{322}(v_L) = 0$ and $N^0_{311}(v_L) \le 0$. Due to formulas (C.38)$_3$, inequality (C.36)$_2$ is satisfied automatically, and therefore the solution of inequality (C.36) is $k \ge 0$.

In Case III, we have $N^0_{311}(v_L) \le -N^0_{322}(v_L) < 0$, and thus inequality (C.36)$_1$ cannot be satisfied. When $\Delta(v_L) < 0$, inequality (C.36)$_2$ holds naturally and as a result the solution of inequality (C.36) is $k \ge 0$; When $\Delta(v_L) \ge 0$, the solution of inequality (C.36)$_2$ is $k < k_2(v_L) \cup k > k_1(v_L)$. By using formulas (C.38) we can prove $k_2(v_L) \le k_1(v_L) \le 0$. Consequently, the solution of inequality (C.36) is $k \ge 0$. Thus we have the conclusion that the solution of inequality (C.36) is $k \ge 0$ in Case III.

In Case IV, we have $-N^0_{322}(v_L) < N^0_{311}(v_L) < 0$. When $\Delta(v_L) < 0$, inequality (C.36)$_2$ holds automatically and consequently the solution of inequality (C.36) is $k \ge 0$. When $\Delta(v_L) \ge 0$, the solution of inequality (C.36) is

$$k > \frac{M_{11}(v_L)}{N^0_{311}(v_L) + N^0_{322}(v_L)} \ \cup\ k < k_2(v_L) \ \cup\ k > k_1(v_L) \tag{C.41}$$

Similarly, by using formulas (C.38) we can prove $k_2(v_L) \le k_1(v_L) < 0$, which means the solution of inequality (C.36) is $k \ge 0$ when $\Delta(v_L) \ge 0$. Thus, in summary, we find the solution of inequality (C.36) is $k \ge 0$ in Case IV.

In Case V, we obtain $N^0_{311}(v_L) = 0$ *and* $N^0_{322}(v_L) \ge 0$. Because inequality (C.36)$_2$ holds automatically in this case, the solution of inequality (C.36) is $k \ge 0$ in this case.

In Case VI, we have $N^0_{311}(v_L) > 0, N^0_{322}(v_L) > 0$ and $\Delta(v_L) > 0$. The solution of inequality (C.36) is

$$k > \frac{M_{11}(v_L)}{N^0_{311}(v_L) + N^0_{322}(v_L)} \ \cup\ k_1(v_L) < k < k_2(v_L) \tag{C.42}$$

By using formulas (C.38) we can prove that $k_1(v_L) < 0$, $k_2(v_L) > 0$ and

$$k_2(v_L) > \frac{M_{11}(v_L)}{N^0_{311}(v_L)} > \frac{M_{11}(v_L)}{N^0_{311}(v_L) + N^0_{322}(v_L)} \tag{C.43}$$

Therefore the solution of inequality (C.36) is $k \ge 0$ in Case VI.

Based on the above discussions, we are able to draw the conclusions presented below: **1)** if the coating film satisfies $\gamma < \lambda_1^{-2}(\lambda_1^2 - \lambda_2^2)$, the solution of $y_1(v_L,k) < 0$ is $0 \le k < k_2(v_L)$; and **2)** if the coating film satisfies $\gamma \ge \lambda_1^{-2}(\lambda_1^2 - \lambda_2^2)$, the solution of $y_1(v_L,k) < 0$ is $k \ge 0$. The corresponding results are summarized in Table C5.

According to the results in Tables C3 and C5, the wavenumber ranges of FRWs corresponding to various biasing fields and various surface coating parameters are concluded and listed in Table 1 in the main text.

2. Existence of SRWs

2.1. Solutions of $y_2(0,k) > 0$

$y_2(0,k) > 0$ is satisfied if either

$$\begin{aligned} &M_{11}(0) + M_{22}(0) - k[N^0_{311}(0) + N^0_{322}(0)] > 0 \\ \text{or}\ \ &H_1(0)k^2 + H_2(0)k + H_3(0) < 0 \end{aligned} \tag{C.44}$$

is satisfied. The analysis is also divided into Groups A and B, and each group is further divided into six different cases respectively as listed in Tables C1 and C2.

2.1.1. Group A. In Cases A1, A2 and A3, we obtain $N^0_{311}(0) + N^0_{322}(0) \le 0$ and consequently inequality (C.44)$_1$ holds automatically. Therefore in these three cases the solution of inequality (C.44) is $k \ge 0$.

In Case A4, we have $-N^0_{322}(0) < N^0_{311}(0) < 0$ and the solution of inequality (C.44) is

$$k < k_{p0} \ \cup\ k < k_2(0) \ \cup\ k > k_1(0) \tag{C.45}$$

in which k_{p0} is defined in Eq. (C.13), $k_1(v)$ and $k_2(v)$ are defined in Eq. (57). Due to inequalities (C.4) and (C.5), $k_1(0) > 0$ and $k_2(0) < 0$. Based on the discussion from formulas (C.14) to (C.19) we have $k_1(0) < k_{p0}$. Therefore the solution of inequality (C.44) is $k \ge 0$ in this case.

In Case A5, we have $N^0_{311}(0) = 0, N^0_{322}(0) > 0$. Then inequality (C.44) degenerates to

$$\begin{aligned} &M_{11}(0) + M_{22}(0) - kN^0_{322}(0) > 0 \\ \text{or}\ \ &N^0_{322}(0)M_{11}(0)k - H_3(0) > 0 \end{aligned} \tag{C.46}$$

Since $N^0_{322}(0) > 0$, the solution of inequality (C.46) is

$$k < k_{p0} \ \cup\ k > k_{p1} \tag{C.47}$$

where k_{p1} is defined in Eq. (C.22). It has been proved that, in Case A5 the relation $k_{p1} < k_{p0}$ is valid. Thus, the solution of inequality (C.46) in Case A5 is $k \ge 0$.

In Case A6, we obtain $N^0_{322}(0) \ge N^0_{311}(0) > 0$. The solution of inequality (C.44) is

$$k < k_{p0} \ \cup\ k_1(0) < k < k_2(0) \tag{C.48}$$

It has been proved that in this case we have $0 < k_1(0) < k_{p0}$ and $k_2(0) > k_{p0} > 0$. Thus, the solution of inequality (C.44) is $0 \le k < k_2(0)$ in Case A6.

2.1.2. Group B. In Case B1, inequality (C.44)$_1$ holds automatically. Therefore the solution of inequality (C.44) is $k \ge 0$ in this case.

In Case B2, we have obtained $\Lambda_1 \le -M_{11}(0)N^0_{322}(0)/M_{22}(0)$ with the help of inequality (C.33). By further using $M_{11}(0) \ge M_{22}(0) > 0$ and the conditions of Case B2 as indicated in Table C2, we obtain $N^0_{311}(0) < -N^0_{322}(0)$. Thus inequality (C.44)$_1$ is satisfied automatically and the solution of inequality (C.44) is $k \ge 0$.

In Case B3, the discriminant of the left-handed quadratic form of (C.44)$_2$ is negative, i.e. $\Delta(0) \le 0$. When $N^0_{311}(0) = \Lambda_1$, we have $\Delta(0) = 0$ and consequently inequality (C.44)$_2$ cannot be satisfied. Due to $\Lambda_1 \le -M_{11}(0)N^0_{322}(0)/M_{22}(0) \le -N^0_{322}(0)$, then inequality (C.44)$_1$ is satisfied automatically with the help of inequality (C.4). Thus when $N^0_{311}(0) = \Lambda_1$, the solution of inequality (C.44) is $k \ge 0$. When $\Lambda_1 < N^0_{311}(0) < \Lambda_2$, we obtain $\Delta(0) < 0$. Thus inequality (C.44)$_2$ holds automatically, and as a result the solution of inequality (C.44) is $k \ge 0$. When $N^0_{311}(0) = \Lambda_2$ and $N^0_{311}(0) + N^0_{322}(0) \le 0$, inequality (C.44)$_1$ is satisfied automatically and the solution of inequality (C.44) is $k \ge 0$. When $N^0_{311}(0) = \Lambda_2$ and $N^0_{311}(0) + N^0_{322}(0) > 0$, the solution of inequality (C.44) is

$$k < k_{po} \ \cup\ k < k_1(0) \ \cup\ k > k_1(0) \tag{C.49}$$

By analyzing formula (C.14) we can prove that $k_1(0) < k_{p0}$ and as a result the solution of inequality (C.44) is $k \ge 0$. According to the above discussions, we find the solution of inequality (C.44) is $k \ge 0$ in Case B3.

In Case B4, when $N^0_{311}(0) + N^0_{322}(0) \le 0$, we find inequality (C.44)$_1$ holds automatically, and consequently the solution of inequality (C.44) is $k \ge 0$; when $N^0_{311}(0) + N^0_{322}(0) > 0$ the solution of inequality (C.44) is

$$k < k_{po} \ \cup\ k < k_2(0) \ \cup\ k > k_1(0) \tag{C.50}$$

With the help of inequalities $N^0_{311}(0) + N^0_{322}(0) > 0$, $M_{11}(0) \ge M_{22}(0)$ and $N^0_{311}(0) < N^0_{322}(0)$, we obtain $k_1(0) - k_{p0} < 0$ by analyzing formula (C.35). Thus the solution of inequality (C.44) is $k \ge 0$ in Case B4.

Table C6
Solutions of $y_2(0,k) > 0$ in all situations.

Conditions	Wavenumber range
$\lambda_1^2 \geq \vartheta_2\lambda_2^2$	$k \geq 0$
$\lambda_1^2 < \vartheta_2\lambda_2^2$	$0 \leq k < k_2(0)$

In Case B5, inequality (C.44) degenerates to inequality (C.46). Due to the condition $N_{322}^0(0) > 0$, inequality $(C.46)_2$ holds automatically. Thus, the solution of inequality (C.46) is $k \geq 0$ in this case.

In Case B6, the discriminant of the left-handed quadratic form of $(C.44)_2$ is positive, i.e. $\Delta(0) > 0$, and the solution of inequality (C.44) is given by formula (C.47). It has been proved that $k_1(0) \leq 0$, $k_2(0) > 0$ and $k_2(0) \geq k_{p0}$ in this case. Thus, the solution of inequality (C.44) is $0 \leq k < k_2(0)$ in Case B6.

2.1.3. Groups A and B. From the above discussions of Groups A and B, we find the solution of $y_2(0,k) > 0$ only depends on $N_{311}^0(0)$: if $N_{311}^0(0) \leq 0$, i.e. $\lambda_1^2 \geq \vartheta_2\lambda_2^2$, the solution of inequality (C.44) is $k \geq 0$; if $N_{311}^0(0) > 0$, i.e. $\lambda_1^2 < \vartheta_2\lambda_2^2$, the solution is $0 \leq k < k_2(0)$. The corresponding solutions are listed in Table C6 for various situations.

2.2. Solutions of $y_2(v_L,k) < 0$

$y_2(v_L,k) < 0$ holds if both

$$\begin{aligned} &M_{11}(v_L) - k[N_{311}^0(v_L) + N_{322}^0(v_L)] < 0, \\ \text{and } &N_{311}^0(v_L)N_{322}^0(v_L)k^2 - N_{322}^0(v_L)M_{11}(v_L)k + M_{12}^2(v_L) > 0 \end{aligned} \quad (C.51)$$

are satisfied. The solutions of inequality (C.51) will be obtained according to the six different cases as listed in Table C4.

In Cases I, II and III, inequality $(C.51)_1$ cannot be satisfied due to $N_{311}^0(v_L) + N_{322}^0(v_L) \leq 0$. Thus, inequality (C.51) has no solution, i.e. $k \in \emptyset$.

In Case IV, when $\Delta(v_L) < 0$, inequality (C.51) has no solution, i.e. $k \in \emptyset$; when $\Delta(v_L) \geq 0$, the solution is

$$k > \frac{M_{11}(v_L)}{N_{311}^0(v_L) + N_{322}^0(v_L)} \ \cap \ k_2(v_L) < k < k_1(v_L) \quad (C.52)$$

It has been proved that in this case $k_2(v_L) \leq k_1(v_L) < 0$, which implies that inequality (C.51) has no solution. As a result, inequality (C.51) has no solution in Case IV, i.e. $k \in \emptyset$.

In Case V, we find $N_{311}^0(v_L) = 0, N_{322}^0(v_L) \geq 0$. Inequality $(C.51)_2$ degenerates to

$$-N_{322}^0(v_L)M_{11}(v_L)k + M_{12}^2(v_L) > 0 \quad (C.53)$$

The above inequality cannot be satisfied because of inequality (C.4). Thus inequality (C.51) has no solution again in Case V.

In Case VI, we find $0 < N_{311}^0(v_L) \leq N_{322}^0(v_L)$ and $\Delta(v_L) > 0$. The solution of inequality $(C.51)_1$ is

$$k > \frac{M_{11}(v_L)}{N_{311}^0(v_L) + N_{322}^0(v_L)} \quad (C.54)$$

and the solution of inequality $(C.51)_2$ is

$$k < k_1(v_L) \ \cup \ k > k_2(v_L) \quad (C.55)$$

It has been proved that $k_1(v_L) < 0$ and $k_2(v_L) > M_{11}(v_L)/[N_{311}^0(v_L) + N_{322}^0(v_L)] \geq 0$ in this case. Thus the solution of inequality (C.51) is $k > k_2(v_L)$ in Case VI.

As a conclusion, inequality (C.51) has no solution from Case I to Case V, and only in Case VI it has the solution as $k > k_2(v_L)$. The corresponding results are listed in Table C7.

By combining Tables C6 and C7, we are able to obtain the wavenumber ranges of SRWs corresponding to various biasing fields and various coating film parameters, which are summarized in Table 2 in the main text.

Table C7
Solutions of $y_2(v_L,k) < 0$ in all situations.

Conditions	Wave number range
$\gamma > 1 - \vartheta_2\lambda_2^2\lambda_1^{-2}$	$k > k_2(v_L)$
$\gamma \leq 1 - \vartheta_2\lambda_2^2\lambda_1^{-2}$	$k \in \emptyset$

Appendix D. List of Symbols

A_{0ijks}, $(i, j, k, s = 1, 2, 3)$	Effective elastic tensor
B_r, B_0, B_t	Reference, initial and current configurations
∂B_r, ∂B_0, ∂B_t	Boundaries in the reference, initial and current configurations
C, D, S, T, s_p, v_p	Parameters defined in Eq. (B.5)
C_{1ij}, C_{2ij}, C_{3ij}, $\hat{C}_{1ij}$, $\hat{C}_{2ij}$, $\hat{C}_{3ij}$, $\breve{C}_{1ij}$, $\breve{C}_{2ij}$, $\breve{C}_{3ij}$	Matrices of effective elastic coefficients, as defined in Eq. (8)
$f(J)$	Response function of the Hadamard material
H, h	Thicknesses of the coating film in the reference and initial configurations
$H_1(v)$, $H_2(v)$, $H_3(v)$	Parameters (depending on the velocity v) defined in Eq. (52)
J	Ratio of the infinitesimal volumes in the two configurations
$\dot{K}_{0ij}$ $(i, j, k, s = 1, 2, 3)$	Incremental nominal stress tensor after 'push forward' operation
k_{p0}, k_{p1}	Parameters defined in Eqs. (C.13) and (C.22)
k, s	Dimensionless wavenumber and velocity
$k_1(v)$, $k_2(v)$	Parameters defined in Eq. (57)
p_i, $p_i(\varphi)$ $(i = 1, 2, \cdots, 6)$	Eigenvalues of $\mathbf{N}$ and $\mathbf{N}(\varphi)$
v_b	Parameter defined as $v_b = \sqrt{\mu/\rho_r}$
v_L	Limiting speed
$y_i(v,k)$ $(i = 1, 2, 3)$	Eigenvalues of $\mathbf{Z}(v, k)$
(X_1,X_2,X_3), (x_1,x_2,x_3)	Cartesian coordinates in the reference and initial configurations
α, β, θ_i	Parameters defined in Eq. (41)
ρ, ρ_r	Mass densities in the current and reference configurations
Ω, Ω^F	Energy density functions of the substrate and thin film per unit volume in B_r
δ_{ij}	Kronecker delta
γ, $\Delta(v)$, Ψ	Parameters defined in Eq. (57)
η	Modulus ratio defined in Eq. (36)
λ_1, λ_2	Pre-stretches
ϑ_2	Parameter defined as $\vartheta_2 = (4\lambda_2^2 - 3\theta_2^2)/\theta_2^2$
Λ_1, Λ_2	Parameters defined in Eq. (C.31)
$\mathbf{a}$, $\mathbf{b}$	Amplitudes of displacement and stress potential vectors
$\mathbf{A}$, $\mathbf{B}$	Eigenvector matrices defined in Eq. (23)
$\mathbf{F}$	Deformation gradient tensor of the pre-deformation
$\mathbf{G}_1^F$	Matrix defined in Eq. (29)
$\mathbf{I}$	Identity matrix
$\tilde{\mathbf{N}}$, $\mathbf{S}$, $\mathbf{H}$, $\mathbf{L}$	Integral matrices defined in Eqs. (21) and (22)
$\mathbf{N}$, $\boldsymbol{\xi}$	Fundamental elastic tensor and state vector
$\mathbf{N}(\varphi)$	Fundamental elastic tensor in the rotated coordinate system
$\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$	Matrices defined in Eq. (18)
$\mathbf{M}(v)$	Impedance matrix
$\mathbf{u}$	Displacement vector of the incremental infinitesimal deformation
$\mathbf{N}_r$, $\mathbf{N}_0$, $\mathbf{n}$	Outward normals in the reference, initial and current configurations
$\mathbf{X}$, $\mathbf{x}$	Position vectors in the reference and initial configurations
$\mathbf{Z}(v, k)$	Matrix defined in Eq. (31)
$\boldsymbol{\Phi}$	Stress potential vector
$\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, $\boldsymbol{\tau}_3$	Incremental nominal stress vectors
$\emptyset$, $\cap$, $\cup$	Null set, intersection of sets, union of sets
Superscript F	Indicating physical quantities/parameters in the coating film

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